Task: Blogs and Frogs | **Algebra II/Core Math III**

George has two problems to consider:

Problem 1: Erin has moved to a new school in the Virgin Islands. To keep her friends up-to-date on her adventures, she started a blog. In the first week, she had five friends following her blog. Her friends thought the blog was interesting, so the next week each of her friends told two additional friends, who began following the blog. In the next week, each of the new followers from the previous week told two of their friends. This continued for several weeks.

a) Make a three-column table to represent this problem. In the first column, list the number of weeks since Erin's blog began. In the second column, tell how many NEW followers Erin's blog has attracted each week. In the last column, give the TOTAL number of followers Erin's blog has.

b) Write an expression to describe how to find the number of NEW followers Erin's blog has in week N.

c) Write an expression to describe the TOTAL number of followers Erin's blog has in week N.

Problem 2: A frog is sitting a fixed distance away from the pond. He starts hopping towards the pond. In the first hop, he jumps 1/3 of the distance between his original position and the pond. In the second hop, he jumps 1/3 of the remaining distance. In the third hop, he jumps 1/3 of the distance that remains after his second hop. This pattern continues for all of the frog's hops.

a) Make a table to represent the frog's progress after each hop. Include a column in your table to show the remaining distance after each hop.

b) Write an expression to describe how to find the remaining distance after the Nth hop.

c) Will the frog ever reach the pond? Why or why not?

George needs your help. Solve both of George's problems. Describe how these problems are similar and how these problems are different.

Teacher Notes:

This is a long task. Teachers may want to assign problem 1 and problem 2 on different days or assign problem 1 to one half of the class and problem 2 to the other half of the class in order to complete the entire task in a shorter amount of time. If teachers take the second option (half of the class works on each problem), then the similarities/differences of the problems should be part of the class whole-group discussion.

Both of these problems are examples of geometric sequences and geometric series. The first problem illustrates a geometric series that keeps growing without bound—in other words, the geometric series "diverges". The second problem illustrates a geometric series that grows, but it grows within a bound—in other words, the geometric series "converges".

Common Core State Standards for Mathematical Content	Common Core State Standards for Mathematical Practice
Write expressions in equivalent forms to solve problems **A-SSE.B.4** Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★ *Modeling standards appear throughout the high school standards indicated by a star symbol (★).*	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Essential Understandings

- The concept of function is intentionally broad and flexible, allowing it to apply to a wide range of situations. The notion of function encompasses many types of mathematical entities in addition to "classical" functions that describe quantities that vary continuously. For example, matrices and arithmetic and geometric sequences can be viewed as functions.
- Functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, and tabular representations. Links among these different representations are important to studying relationships and change.
- The convergence or divergence of the geometric series depends on the value of the common ratio.

Explore Phase

Possible Solution Paths	Assessing and Advancing Questions
Problem 1: Part (a): In Week 1, Erin gets her original 5 friends to follow her blog, so the total number of followers is 5. In Week 2, each of the 5 followers from week 1 tells 2 friends, so the number of new followers is 5 x 2 = 10. The total number of followers is then 5 + 10 = 15. In Week 3, each of the 10 new followers from week 2 tells 2 friends,	**Assessing questions:** How did you calculate the number of new followers in each week? How did you calculate the total number of followers? **Advancing questions:** What is the difference in "new followers" and "total number of followers"?

so the number of new followers in week 3 is 10 x 2 = 20. The total number of followers is: 5 (from week 1) + 10 (from week 2) + 20 (from week 3) = 35.

This pattern continues, so your table should look like the following:

Week	New Followers	Total Followers
1	5	5
2	10	5 + 10 = 15
3	20	15 + 20 = 35
4	40	35 + 40 = 75

How can you determine how many new followers Erin's blog has?

Problem 1: Part (b): We can use the table in part (a) to establish a pattern.

Week	New Followers	Pattern
1	5	5
2	10	10 = 5 x 2
3	20	20 = 10 x 2 = 5 x 2 x 2 = 5 x 2^2
4	40	40 = 20 x 2 = 5 x 2 x 2 x 2 = 5 x 2^3

In general, if the student's table begins with week 1, then the pattern for each of the entries in the "New Followers" column is 5 x 2^{power}. The power on 2 is one less than the week number, so our expression is:

In week N, the number of new followers will be 5 x 2^{N-1}.
(We can write this as f(N) = 5 x 2^{N-1}.)

Assessing questions:

What does your variable in the exponent represent? How can you be sure your formula is accurate?

Advancing questions:

What happens to the number of new followers each week? How can you use this information to describe a pattern?

Problem 1: Part (c): Students should recognize that the number of

Assessing questions:

<table>
<tr>
<td>new followers is a geometric sequence and the total number of followers can be found using the sum of a geometric series.

Approach 1: At a minimum, students should recognize that the sum (assuming that the table in part (a) begins with week 1) is:
$5 + 10 + 20 + 40 + \ldots + 5 \times 2^{N-1}$.

Approach 2: Once students understand that the total number of followers can be found using the expression in Approach 1, they may either recognize this as the sum of a geometric series and immediately write this using a compact form:

$5 + 10 + 20 + 40 + \ldots + 5 \times 2^{N-1} = (5 - 5(2)^{N-1}) / (1 - 2)$
$= 5\,(2)^N - 5.$

Approach 3: Students who do not recognize this as the sum of a geometric series can still find the compact form by deriving the formula. To derive the formula, students will need to recognize that if you multiply each term of your sum by 2, you will get the next term in your sum (except when you multiply the last term by 2—then you “create” a new term). You can take advantage of this fact:

Let $S = 5 + 10 + 20 + 40 + \ldots + 5 \times 2^{N-1}$
Then $2S = 10 + 20 + 40 + \ldots + 5 \times 2^{N-1} + 5 \times 2^N$

If we subtract the second equation from the first equation, we have:
$S - 2S = 5 - 5\,(2)^N$
or $-S = 5 - 5\,(2)^N$
or $S = 5\,(2)^N - 5.$</td>
<td>How do you know your formula works?

What does the variable represent in your formula?

Advancing questions:
What kind of sequence are you examining in this problem? How does knowing the type of sequence in the problem help you?</td>
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<td>Problem 2: Part (a): The most difficult part of this problem may be the fact that the distance between the frog and the pond is not given. Student may elect to assign a distance (say, for example, 300 feet) then use the distance they have assigned to begin their</td>
<td>Assessing questions:
What effect does not knowing the original distance between the frog and the pond have on your calculations?</td>
</tr>
</table>

calculations (so, in the first hop, the frog will have covered 100 feet and have 200 feet remaining). In the solutions provided, the distance between the frog's initial position and the pond will be 1 unit. This will enable teachers to "adjust" to fit whatever distance the student assigns by multiplying the distances in the solutions by the student's choice of initial distance.

Approach 1: Students may calculate the total distance covered by adding 1/3 of the remaining distance to each of the distances covered in the table.

Hop	Distance Covered	Distance Left
1	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}\right)=\frac{5}{9}$	$\frac{4}{9}$
3	$\frac{5}{9}+\frac{1}{3}\left(\frac{4}{9}\right)=\frac{19}{27}$	$\frac{8}{27}$

Approach 2: Note that the distance left in the table above forms a nice pattern: the distance left can be found using

$$\left(\frac{2}{3}\right)^N,$$

where N represents the number of hops. Then the distance covered can be calculated using

$$1-\left(\frac{2}{3}\right)^N.$$

This information may be used to fill in the table.

Explain how you determined the distance the frog had hopped after N hops.

Advancing questions:
How far is the frog from the pond? What can you do to work the problem if this distance is not known?

Would a picture help you understand what is happening in the problem?

Problem 2: Part (b): The distance left can be found using

$$\left(\frac{2}{3}\right)^N,$$

where N represents the number of hops. This can be calculated from the table:

Hop	Distance Left
1	$\frac{2}{3}$
2	$\frac{4}{9}=\left(\frac{2}{3}\right)^2$
3	$\frac{8}{27}=\left(\frac{2}{3}\right)^3$

Assessing questions:
What does the variable in your formula represent?

How do you know that your formula works?

Advancing questions:
Using your table from part (a), do you see a pattern for the distance left?

How are the numbers in the "distance left" column related to each other?

Problem 2: Part (c): Mathematically, the frog will not reach the pond after a finite number of hops. There are several arguments that can be used to support this.

Argument 1: The distance between the frog and the pond after N hops was found in Problem 2, part (b):

The distance left can be found using

$$\left(\frac{2}{3}\right)^N,$$

where N represents the number of hops.

Since there is always a positive distance left, mathematically the frog will never reach the pond.

Argument 2: The distance the frog has travelled can be found in three different ways. First, we can use the distance left presented in Problem 2, part (b) above and subtract this distance from 1 (the

Assessing questions:
Mathematically, do you think the frog will ever reach the pond?

Is there a difference between a mathematical answer and a "common sense" answer? How is your mathematical answer different from your "common sense" answer?

Advancing questions:
Do you think the frog will ever reach the pond? Does your answer depend on how far away from the pond the frog begins? How could you support your answer?

total distance that must be covered): The distance traveled is $$1-\left(\frac{2}{3}\right)^N.$$ Second, students may recognize that the distance traveled is a geometric series and use the formula for summing a geometric series: In this geometric series, $a_1 = \frac{1}{3}$ and $r = \frac{2}{3}$ so the sum of the first n terms is $$S = \frac{\frac{1}{3}-\frac{1}{3}\left(\frac{2}{3}\right)^N}{1-\frac{2}{3}} = 1-\left(\frac{2}{3}\right)^N.$$ Third, students may try to generate the sum of the geometric series using a process similar to that in Problem 1, part (c): Let S represent the sum of the first N distances hopped by the frog. Then $S = \frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\right)^2+\frac{1}{3}\left(\frac{2}{3}\right)^3+\ldots+\frac{1}{3}\left(\frac{2}{3}\right)^{N-1}$ $$\left(\frac{2}{3}\right)S = \frac{1}{3}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\right)^2+\frac{1}{3}\left(\frac{2}{3}\right)^3+\ldots+\frac{1}{3}\left(\frac{2}{3}\right)^N$$ If we subtract the second equation from the first equation, we have: $$S-\left(\frac{2}{3}\right)S = \frac{1}{3}-\frac{1}{3}\left(\frac{2}{3}\right)^N$$	

$\left(\frac{1}{3}\right)S = \frac{1}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^N$ Multiplying both sides of this equation by 3 gives us: $S = 1 - \left(\frac{2}{3}\right)^N$ Since there is always a positive distance left, the frog will never reach the pond. Note: As N gets larger and larger (in mathematics we talk about "as N approaches infinity), note that the distance left gets smaller and smaller (we say that the term being subtracted "approaches 0"), so that the frog gets closer and closer to completing the distance. In calculus these ideas lead to limits. (This may spark a conversation among your students regarding the fact that eventually, the frog will get close enough to the pond that he can simply step over into the pond because the distance left is so small.)	
Possible Student Misconceptions	
In the blog problem, students may not understand the difference between the total number of followers and the number of new followers. For example, students could follow this line of reasoning: Week 1: There are 5 followers. Week 2: Each of these 5 told 2 friends, so we have a total of 15 followers. Week 3: Each of the 15 followers told 2 friends, so we have a total of 15 + 30 = 45 followers. Week 4: Each of the 45 followers told 2 friends, so we have a total of 45 + 90 = 135 followers. Etc.	Read the problem carefully. Explain to me how many followers you would have in each of weeks 1, 2, 3, and 4. How did you calculate the number of new followers each week? Does this reflect the directions in the problem?

In this case, the students are assuming that ALL of the followers from the previous week are telling 2 friends, so they are doubling the TOTAL number of followers, not doubling the total number of NEW followers.	
In the frog problem, students may calculate each successive hop as being 1/3 of the previous hop rather than 1/3 of the distance left. This reasoning will work if the ratio is ½ but not if the ratio is any other fraction.	Draw a picture to show me how the frog is progressing. How far does the frog jump on his first hop? What distance is left? How does that affect the distance the frog travels on his second hop?
Entry/Extensions	**Assessing and Advancing Questions**
If students can't get started….	What information does the problem ask you to put in your table? How does making a table help you organize your information? Do you see any patterns in your table?
If students finish early….	Re-work the frog problem using different values for the common ratio. Is there a reasonable value of the common ratio that will allow your frog to reach the pond in a finite number of hops? Why or why not? In the blog problem, are there "real-world" limitations to the number of followers Erin's blog can have? If so, what are those limitations?
Discuss/Analyze	
Whole Group Questions	
What happens to the number of blog followers in the first problem as N gets bigger and bigger? What happens to the distance the frog travels in the second problem as N gets bigger and bigger? How are these problems alike? How are they different? What is the key piece of information in the problem that determines whether your sum diverges (as in the blog problem) or converges (as in the frog problem)?	

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