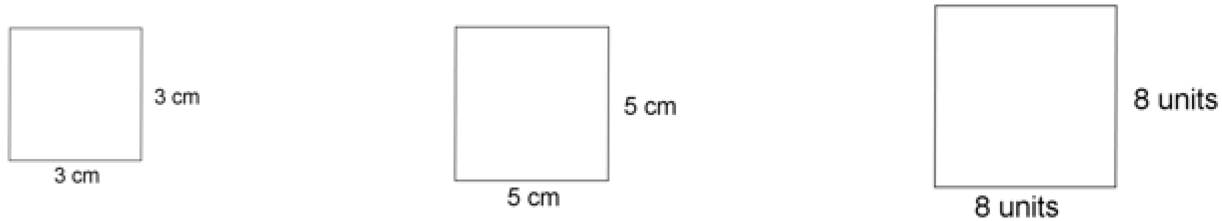
Task: Ratios, Proportions, and Similar Figures **Geometry/Core Math II**

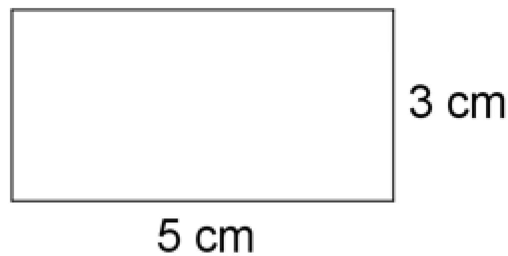
1. a) Each of the following figures is a square. Calculate the length of each diagonal. Do not round your answer.



b) What do you notice about the length of the diagonal for each of these squares? Write a general rule for finding the length of the diagonal of a square if you know the length of the side <u>WITHOUT DOING ANY CALCULATIONS</u>.

c) What is the ratio of the length of the side of a square to the length of its diagonal? Is this ratio the same for all squares? Why or why not?

2. a) Calculate the length of the diagonal of the rectangle. Do not round your answer.



b) What is the ratio of the length of the shortest side of this rectangle to the length of the diagonal?

c) Find two other rectangles that have the same ratio of the length of the shortest side to the length of the diagonal.

d) How are the rectangles you found in part (c) related to the rectangle in part (a)? Make a conjecture about all rectangles that have the same ratio of the length of the shortest side to the length of the diagonal.

e) Prove your conjecture in part (d).

3. a) Construct a rectangle with diagonal length 2 such that the angle between the diagonal and the shortest side is 60^{o}. What is the ratio of the length of the shortest side to the length of the diagonal?

b) Draw two other rectangles with the same ratio of the length of the shortest side to the length of the diagonal as your answer in part (a).

What is the measure of the angle between the diagonal and the shortest side?

c) Make a conjecture about the relationship between the 60^{o} angle between the diagonal and the shortest side and the ratio of the length of the shortest side to the length of the diagonal.

d) Prove your conjecture in part (c).

e) If you draw two different rectangles with the same angle between the diagonal and the shortest side, what can you say about the ratio of the length of the shortest side to the length of the diagonal? Why?

Teacher Notes:

This is a long task. Teachers may want to use problems 1, 2, and 3 on different days. Problem 1 may also be skipped in the interest of time—the main purpose of problem 1 is to set up the student mindset of the ratio being the same regardless of the size of the square.

Students will need to use the concept of similar figures and the Pythagorean Theorem to complete many of these problems. Students will also need a ruler and protractor to complete problem 3.

Geometry software such as Geogebra (www.geogebra.org) would be helpful for some parts (but it is not necessary to have access to geometry software to complete the task).

Common Core State Standards for Mathematical Content	Common Core State Standards for Mathematical Practice
Define trigonometric ratios and solve problems involving right triangles G-SRT.C.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.	Mathematical Practices 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Essential Understandings

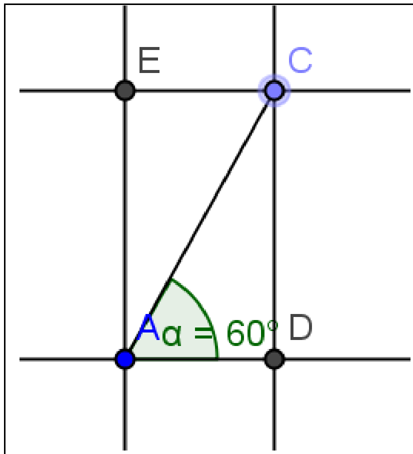
- Geometry is about working with variance and invariance, despite appearing to be about theorems.
- Underlying any geometric theorem is an invariance—something that does not change while something else does.
- Invariances are rare and can be appreciated only when they emerge out of much greater variation.
- Examining the possible variations of an invariant situation can lead to new conjectures and theorems.

Explore Phase	
Possible Solution Paths	**Assessing and Advancing Questions**
1a) Students should use the Pythagorean Theorem to find the length of each diagonal. The calculation for the first square is given below; the calculations for the other squares are similar. For the first square, the diagonal will divide the square into two right triangles. The lengths of the legs of one of these right triangles is 3 cm. Thus, using $a^2 + b^2 = c^2$: $3^2 + 3^2 = c^2$ (where c is the length of the diagonal) $9 + 9 = c^2$ $18 = c^2$ $c = \sqrt{18} = 3\sqrt{2}$ cm Similarly, for the second square, $c = 5\sqrt{2}$ cm and for the third square, $c = 8\sqrt{2}$ units.	**Assessing Questions:** Why did you use the Pythagorean Theorem? How did you know that you could use the Pythagorean Theorem to solve this problem? **Advancing Questions:** Draw one diagonal on the square. What does the diagonal do to the square? Do you see any shapes other than the square in your figure? What kind of triangle do you have?
1b) The length of the diagonal of a square is equal to the length of the side times $\sqrt{2}$.	**Assessing Question:** How do you know this is true for ALL squares? **Advancing Question:** Look at the answers you got in part (a). Do you see any patterns?
1c) The ratio of the length of the side of a square to the length of its diagonal is: $\frac{length\,of\ side}{(length\,of\ side)\sqrt{2}} = \frac{1}{\sqrt{2}}$. The ratio is the same for all squares because the length of the diagonal of a square is equal to the length of the side of the square times $\sqrt{2}$.	**Assessing Questions:** How do you know your ratio is true for ALL squares? Is your ratio true for rectangles that are not squares? Why or why not? **Advancing Questions:** What does “ratio” mean?

	How do you know which value to put in the numerator and which to put in the denominator of your ratio?
2a) Students should use the Pythagorean Theorem to find the length of the diagonal. The diagonal will divide the rectangle into two right triangles. The length of one of the legs is 3 cm and the length of the other leg is 5 cm for each triangle. Thus, using $a^2 + b^2 = c^2$: $3^2 + 5^2 = c^2$ $9 + 25 = c^2$ $34 = c^2$ $c = \sqrt{34}$ cm.	**Assessing Questions:** Why did you use the Pythagorean Theorem? How did you know that you could use the Pythagorean Theorem to solve this problem? **Advancing Questions:** Draw one diagonal on the square. What does the diagonal do to the square? Do you see any shapes other than the square in your figure? What kind of triangle do you have?
2b) The ratio of the length of the shortest side of the rectangle (3 cm) to the length of the diagonal ($\sqrt{34}$ cm) is $\frac{3}{\sqrt{34}}$.	**Assessing Question:** How do you know which value to put in the numerator and which to put in the denominator of your ratio? **Advancing Question:** What does "ratio" mean?
2c) Answers will vary. However, all answers should represent rectangles that are similar to the original rectangle given in part (a). (In other words, the sides of the rectangles drawn by the students should be scalar multiples of the sides of the original rectangle—3 cm by 5 cm. Units may vary.) Examples of rectangles that can be used are: 6 cm by 10 cm 30 cm by 50 cm 0.3 units by 0.5 units 15 units by 25 units	**Assessing Questions:** How did you decide the dimensions of your rectangles? How are the rectangles you have found related to the original rectangle and to each other? Are there other rectangles you could draw that would have the same ratio? **Advancing Questions:**

	How are ratios related to fractions? Can you use your knowledge of fractions to help you find a rectangle with the ratio you need?
2d) All rectangles found in part (c) are similar to the original rectangle given in part (a). Conjecture: Given a rectangle with a fixed ratio of the length of the shortest side to the length of the diagonal, any other rectangle with the same fixed ratio must be similar to the original rectangle.	**Assessing Questions:** How did you decide what your conjecture should be? **Advancing Questions:** Do you see any relationships between the rectangles you found in part (c) and the original rectangle in part (a)? Can you generalize what you did in part (c)?
2e) Conjecture: Given a rectangle with a fixed ratio of the length of the shortest side to the length of the diagonal, any other rectangle with the same fixed ratio must be similar to the original rectangle. Proof: Suppose we are given a rectangle with the length of the short side represented by A and the length of the diagonal represented by C. Then the ratio of the length of the shortest side to the length of the diagonal is $\frac{A}{C}$. To find the length of the long side, we can use the Pythagorean Theorem: $A^2 + b^2 = C^2$ $b^2 = C^2 - A^2$ $b = \sqrt{C^2 - A^2}$. So our original rectangle has the shortest side given by A and the longest side given by $\sqrt{C^2 - A^2}$, where C is the length of the diagonal. Suppose we have a second rectangle whose ratio of the length of the shortest side to the length of the diagonal is $\frac{A}{C}$. Then the	**Assessing Questions:** How did you use the idea of "similarity" in your proof? How do you know the ratios are the same for all these rectangles? How does the ratio being the same force the rectangles to be similar? **Advancing Questions:** How do you know your conjecture is true? On what evidence did you base your theory? What is it that you need to demonstrate?

length of the shortest side must be a scalar multiple of A and the length of the diagonal must be the same scalar multiple of C, where the scalar is greater than 0. In symbols, let $m > 0$ represent the scalar. Then the length of the shortest side is mA and the length of the diagonal is mC. Using the Pythagorean Theorem we can find the length of the longest side of the rectangle: $(mA)^2 + b^2 = (mC)^2$ $b^2 = (mC)^2 - (mA)^2$ $b^2 = m^2 (C^2 - A^2)$ $b = m\sqrt{C^2 - A^2}$. Our second rectangle has the shortest side given by mA and the longest side given by $m\sqrt{C^2 - A^2}$, where mC is the length of the diagonal. Thus our second rectangle's dimensions are a scalar multiple of the first rectangle's dimensions. Since all of the angles of a rectangle measure $90°$, the second rectangle must be similar to the first rectangle by the definition of similar figures.	
3a) In the figure below, rectangle ADCE is constructed so that the length of the diagonal is 2 units and the angle between the diagonal and the shortest side is $60°$. The length of the shortest side is exactly half of the length of the diagonal, so the length of the shortest side is 1 unit. (Note: The diagram was constructed using Geogebra software, available at www.geogebra.org.) The ratio of the length of the shortest side to the length of the diagonal is $\frac{1}{2}$.	**Assessing questions:** How do you know that the figure you constructed is a rectangle? How did you determine the ratio? **Advancing Questions:** How did you draw your triangle? How did you determine the length of the shortest side of your rectangle?



3b) This question is analogous to question 2c. Answers will vary. However, all answers should represent rectangles that are similar to the original rectangle found in part (a). (In other words, the sides of the rectangles drawn by the students should be scalar multiples of the sides of the rectangle in part (a). Units may vary.)

Examples of rectangles that can be used are:

Length of Diagonal	Length of Short Side
3 units	1.5 units
10 units	5 units

Once students have these rectangles drawn, they should find that the angle between the shortest side and the diagonal is always 60° for this ratio.

Assessing Questions:

How did you decide the dimensions of your rectangles?

How are the rectangles you have found related to the original rectangle and to each other?

Are there other rectangles you could draw that would have the same ratio?

Do you see any patterns in the measures of your angles?

Advancing Questions:

How are ratios related to fractions? Can you use your knowledge of fractions to help you find a rectangle with the ratio you need?

3c) The ratio between the shortest side of a rectangle and its diagonal is $\frac{1}{2}$ if and only if the angle between the shortest side of the rectangle and its diagonal is 60°.

Assessing Questions:

How did you decide what your conjecture should be?

Advancing Questions:

Do you see any relationships between the rectangles you found in parts (a) and (b)?

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Can you generalize what you did in part (b)?

What is the relationship between the rectangles you have drawn and the measure of the angle between the short side of the rectangle and the diagonal?

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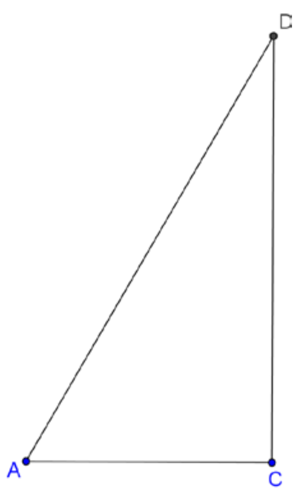
3d) Suppose that in a rectangle, the ratio of the length of the short side to the length of the diagonal is $\frac{1}{2}$. Then the length of the short side of the rectangle is exactly one half of the length of the diagonal. In the drawing below, the right triangle formed by the diagonal (segment AD), one short side (segment AC), and one long side (segment CD) of the rectangle is shown.



Reflect this triangle over line segment CD (see below).

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Assessing Questions:

How do you know the ratios are the same for all these rectangles?

How did you use equilateral triangles in your proof?

(Note: The conjecture is an "if and only if" conjecture—meaning that the proof should have two parts:

(i) the ratio being ½ implies that the measure of the angle is $60°$; and
(ii) the measure of the angle being $60°$ implies that the ratio is ½.

Students may get one part of the conjecture without getting the other part. Teachers may want to spend some time working with the wording of the conjecture to bring out both directions of the proof.)

Advancing Questions:

How do you know your conjecture is true? On what evidence did you base your theory?

What is it that you need to demonstrate?

The right triangle you form inside the rectangle using the diagonal is "special" because of the measures of the angles. Do you know of any other triangles that have one or more of these "special" angles? Can you use those triangles in your proof?

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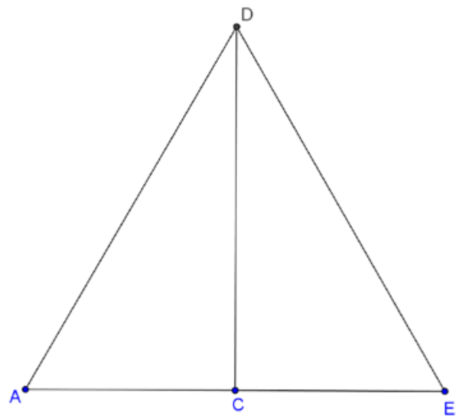
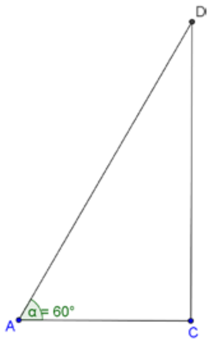
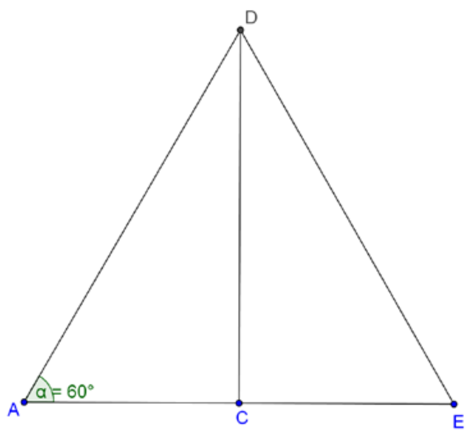


Since triangle ECD is the reflection of triangle ACD, we know that triangle ECD is congruent to triangle ACD, so in particular segment AD is congruent to segment ED and segment AC is congruent to segment EC. We also know that segment AC represents the "short side" of the original rectangle, so the length of segment AC is one half of the length of segment AD (the diagonal of the original rectangle). That means:

the length of segment AC + length of segment CE
= 2 (length of segment AC) (because these lengths are equal)
= 2 (one half of the length of segment AD)
= the length of segment AD.

Thus, the lengths of all three sides of the "big triangle" ADE are equal to each other, so triangle ADE is an equilateral triangle. That means that the measure of angle DAC is $60°$. Therefore the measure of the angle between the short side of the original rectangle and the diagonal is $60°$.

Conversely, suppose the angle between the diagonal and the short side of the rectangle is $60°$. (Only the right triangle formed by the diagonal and one half of the rectangle is shown below.) Note that

the measure of angle ADC is now 30°.  Reflect the triangle over the “long side” given by segment DC.  Since the measure of angle CAD is 60° and the measure of angle ADC is 30°, when we reflect triangle ADC over line segment DC, we must have the measure of angle CED equal to 60° and the measure of angle EDC equal to 30°. That means that the measure of angle ADE is also 60°, making triangle ADE an equilateral triangle. Since triangle ADE is equilateral, we know that the measure of segment AD = the measure of segment DE = the measure of segment AE. We also know that the measure of segment AC = the measure of segment CE (since triangle ADC was reflected over line segment DC, making triangle ADC congruent to triangle EDC). Thus the measure	

of segment AC is one half of the measure of segment AD. Since segment AC is the “short side” of our original rectangle and segment AD is the “diagonal” of our original rectangle, the ratio of the length of the short side to the length of the diagonal of our original rectangle is $\frac{1}{2}$.	
3e) If two different rectangles have the same angle between the diagonal and the shortest side, then the two rectangles must be similar figures. This in turn implies that ratios of the length of the shortest side to the length of the diagonal must be equal using arguments similar to those found in parts 2d and 2e. In the drawing below, rectangle ACED (in black) and rectangle AGFH (in red) have the same angle (shown in green) between the diagonal and the shortest side. In particular, triangle AGF and triangle ACE are both right triangles (each represents half of a rectangle, thus the right angle is “inherited” from the rectangle). By the angle-angle property of similar triangles, this means that triangle AGF and triangle ACE are similar triangles, implying that segment AG is a scalar multiple of segment AC and segment GF is the same scalar multiple of segment CE. This in turn implies that the rectangles are similar figures.	**Assessing Questions:** How did you decide what your conjecture should be? How do you know your conjecture is true? On what evidence did you base your theory? **Advancing Questions:** Can you generalize what you did in part (c)? What happens if you change the size of the angle in part (a)? Can you have two rectangles with the same angle between the diagonal and the short side such that the two rectangles are NOT similar? Why or why not?
Possible Student Misconceptions	
Students may mix up the “shortest side” and the “longest side”.	Ask students to point out the shortest side and the diagonal on their drawings.
Students may not be able to visualize the right triangle formed by two adjacent sides of the rectangle and the diagonal.	Use another sheet of paper to “trace” the triangle formed.
Entry/Extensions	**Assessing and Advancing Questions**
If students can’t get started….	**Assessing Questions:** Why did you use the Pythagorean Theorem? How did you know that you could use the Pythagorean Theorem to solve this problem?

	Advancing Questions: Draw one diagonal on the square. What does the diagonal do to the square? Do you see any shapes other than the square in your figure? What kind of triangle do you have?
If students finish early….	Ask students to go back through problems 2 and 3 using the longest side instead of the shortest side of the rectangle. Do the same types of conjectures hold? Why or why not? Ask students to compute the ratio of the length of the shortest side to the length of the longest side for each rectangle they constructed in problems 2 and 3. Do the same types of conjectures hold for this ratio? Why or why not?

Discuss/Analyze

Whole Group Questions

Understanding 1: In a rectangle, the angle between the diagonal and the short side (or the long side, for that matter) is always between $0°$ and $90°$. This angle does not change if the lengths of the short side and the long side are multiplied by a positive scalar (i.e., if the rectangles are similar).
Questions: What patterns did you see in your rectangles in part 3b? Would these patterns hold if you changed the measure of the angle?

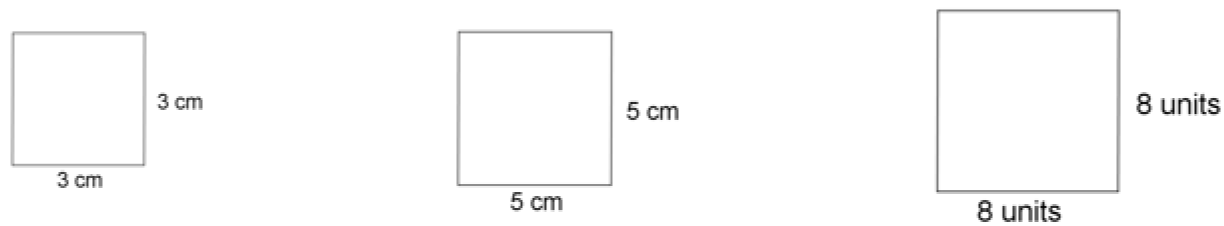
Can you make a conjecture about what would happen in part 3b if you changed the angle to another measure? How would similar rectangles support your conjecture?

Understanding 2: If an angle between $0°$ and $90°$ is fixed, and a rectangle is constructed so that the angle between the diagonal and the short side has the same measure as this fixed angle, then the ratio of the length of the short side to the length of the diagonal is invariant (i.e., since all rectangles that have this fixed angle are similar to each other, these ratios will always be the same). (This understanding leads to the definition of the sine of an angle and the cosine of an angle.)
Questions: What conjectures did you make in part 3e? What patterns did you see?

How can you prove your conjectures in part 3e?

Ratios, Proportions, and Similar Figures

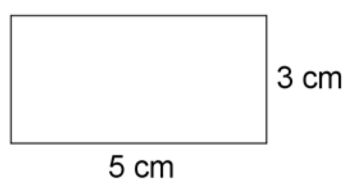
1. a) Each of the following figures is a square. Calculate the length of each diagonal. Do not round your answer.



   b) What do you notice about the length of the diagonal for each of these squares? Write a general rule for finding the length of the diagonal of a square if you know the length of the side WITHOUT DOING ANY CALCULATIONS.

   c) What is the ratio of the length of the side of a square to the length of its diagonal? Is this ratio the same for all squares? Why or why not?

2. a) Calculate the length of the diagonal of the rectangle. Do not round your answer.



   b) What is the ratio of the length of the shortest side of this rectangle to the length of the diagonal?

   c) Find two other rectangles that have the same ratio of the length of the shortest side to the length of the diagonal.

   d) How are the rectangles you found in part (c) related to the rectangle in part (a)? Make a conjecture about all rectangles that have the same ratio of the length of the shortest side to the length of the diagonal.

   e) Prove your conjecture in part (d).

3. a) Construct a rectangle with diagonal length 2 such that the angle between the diagonal and the shortest side is 60^o. What is the ratio of the length of the shortest side to the length of the diagonal?

   b) Find two other rectangles with the same ratio of the length of the shortest side to the length of the diagonal as your answer in part (a). What is the measure of the angle between the diagonal and the shortest side?

   c) Make a conjecture about the relationship between the 60^o angle between the diagonal and the shortest side and the ratio of the length of the shortest side to the length of the diagonal.

   d) Prove your conjecture in part (c).

   e) If you draw two different rectangles with the same angle between the diagonal and the shortest side, what can you say about the ratio of the length of the shortest side to the length of the diagonal? Why?