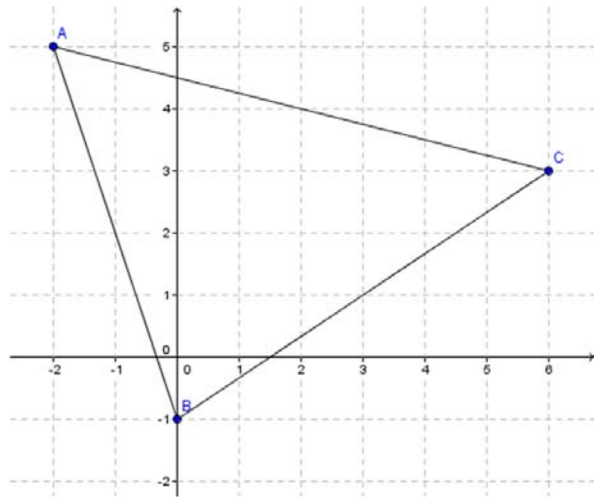
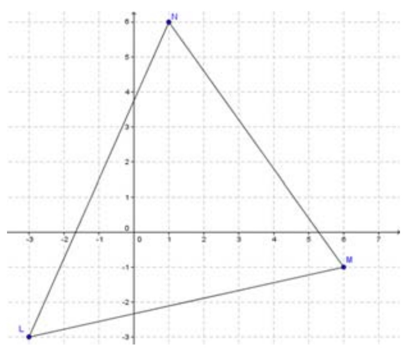
Task: Midpoint Madness | **Geometry/Core Math III**



a) In triangle ABC, choose two sides and find the midpoint of each side. Label the midpoints on your graph. DO NOT FIND THE MIDPOINT OF THE THIRD SIDE.

b) Draw the line segment connecting your midpoints from part (a). Compare this line segment to the third side of the triangle (the one without the midpoint marked). Make two conjectures about the relationship between the third side of the triangle and the line segment connecting the midpoints.



c) Repeat parts (a) and (b) on triangle LMN. Are your conjectures from part (b) true for this triangle? Explain how you know.

d) Are your conjectures true for every triangle? To find out, suppose triangle DUG is drawn on a coordinate plane with D located at point (d_1, d_2), U located at point (u_1, u_2), and G located at point (g_1, g_2). Write your conjectures in terms of D, U, and G, then prove your conjectures using these points.

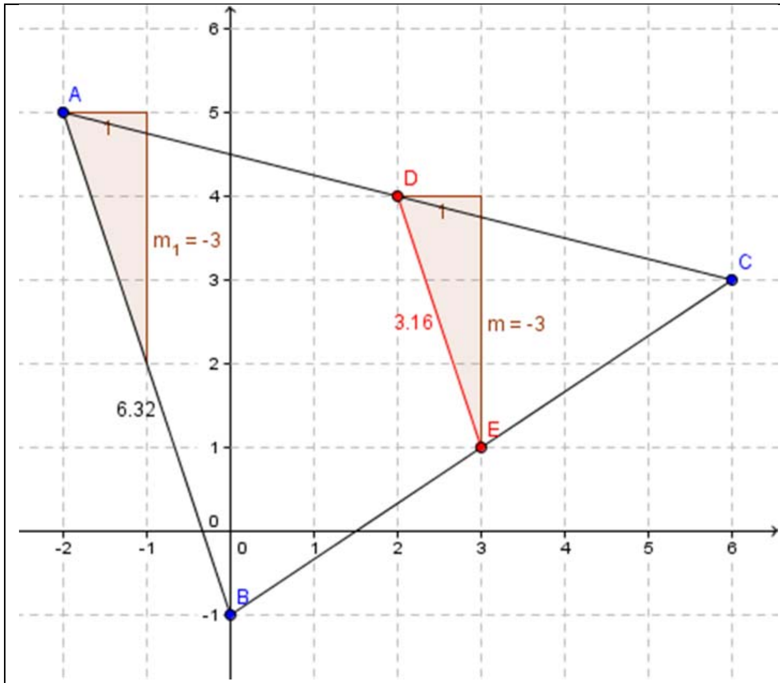
Teacher Notes:	
This task can be used to introduce the idea of the mid-segment of a triangle. The relationship between a given mid-segment and the (unused) third side of a triangle is explored.	
Common Core State Standards for Mathematical Content	**Common Core State Standards for Mathematical Practice**
G-CO.C.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the mid-segments of a triangle meet at a point.* **G-GPE.B.4** Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).* **G-GPE.B.5** Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

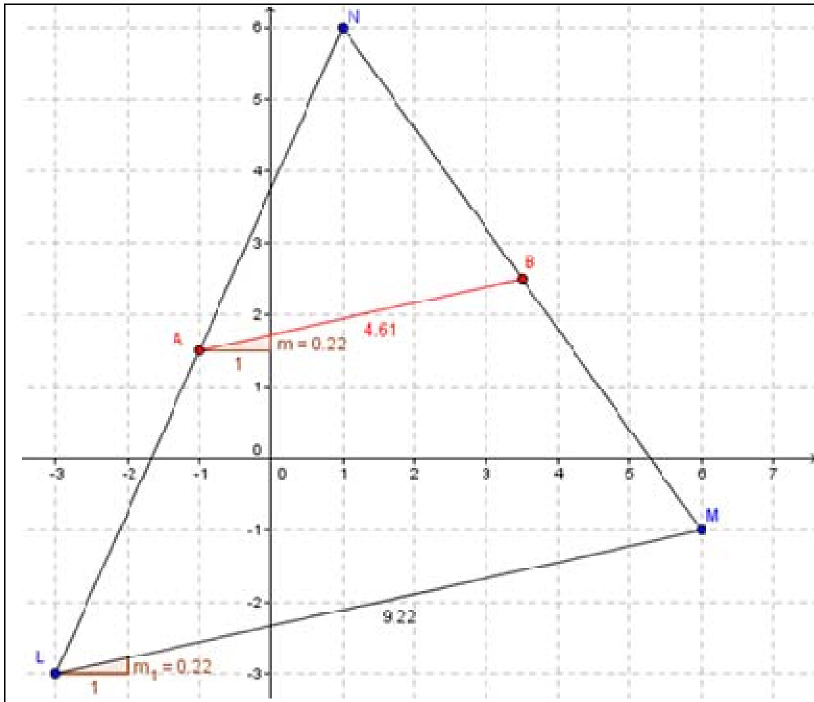
Essential Understandings

- Working with diagrams is central to geometric thinking.
- A diagram is a sophisticated mathematical device for thinking and communicating.
- A diagram is a "built" geometric artifact, with both a history—a narrative of successive construction—and a purpose.
- Underlying any geometric theorem is an invariance—something that does not change while something else does.
- Empirical verification is an important part of the process of proving, but it can never, by itself, constitute a proof.

Explore Phase

Possible Solution Paths	**Assessing and Advancing Questions**
Parts (a) and (b): Students should choose two sides of the triangle, find the midpoints, and draw the line segment connecting the midpoints. In the example below, the midpoints of segments AC and BC are chosen (marked as points D and E respectively), and the segment drawn. (This segment is the mid-segment.) In this particular example, the midpoints can be found either by counting or by calculating the mean x- and y-values of the endpoints of each segment. Conjectures: (1) The mid-segment is parallel to the third side. (2) The length of the mid-segment is half of the length of the third side. To support conjecture (1), students can calculate the slope of each line segment. To support conjecture (2), students may use the distance formula to determine the distance between the endpoints of each segment. The length and the slope are shown in the drawing below. Similar results will be obtained if students choose any two of the three sides.	**Assessing Questions:** How did you find the midpoints of your two sides of the triangle? Why do you think your conjectures are true? **Advancing Questions:** What does "midpoint" mean? How can you find the midpoint of one side? Does the segment connecting your midpoints look similar to the third side of the triangle? How? How can you compare these segments?

	
Part (c): Students should choose two sides of the triangle, find the midpoints, and draw the line segment connecting the midpoints. In the example below, the midpoints of segments LN and MN are chosen (marked as points A and B respectively), and the segment drawn. In this particular example, the midpoints can be found by calculating the mean x- and y-values of the endpoints of each segment. Both conjectures from part (b) hold for this triangle. To support conjecture (1), students can calculate the slope of each line segment. To support conjecture (2), students may use the distance formula to determine the distance between the endpoints of each segment. The length and the slope are shown in the drawing below. (Note that calculations are a little more difficult for this triangle.) Similar results will be obtained if students choose any two of the three sides.	**Assessing Questions:** How did you find the midpoints of your two sides of the triangle? Why do you think your conjectures are true? **Advancing Questions:** What does "midpoint" mean? How can you find the midpoint of one side? Does the segment connecting your midpoints look similar to the third side of the triangle? How? How can you compare these segments?



Part (d): For the proofs of the conjectures, midpoints of segments DU and segments UG are found. The segment connecting these midpoints (labeled A and B respectively in the diagram) is compared to segment DG, the third side of the triangle.

Proof of Conjecture 1: The mid-segment is parallel to the third side.

Part (i): Assume that for points D and G, $d_1 \neq g_1$ (in other words, D and G do not lie on the same horizontal line). Let A represent the midpoint of segment DU and let B represent the midpoint of segment UG. Then:

$$A = \left(\frac{d_1+u_1}{2}, \frac{d_2+u_2}{2}\right) \text{ and } B = \left(\frac{u_1+g_1}{2}, \frac{u_2+g_2}{2}\right).$$

Assessing Questions:

Explain your proof to me. How does your diagram support your proof?

What happens if D and G lie on the same horizontal line? Will your calculations work? (Choose the endpoints of the student's "third side" of the triangle when you ask this question.)

Advancing Questions:

What were your conjectures in part (b)?

How did you show that your conjectures were true for part (c)?

Can you use the same ideas from part (c) to prove your conjectures here?

The slope of segment AB (the mid-segment) is:

$$m_{AB} = \frac{\frac{u_2+g_2}{2} - \frac{d_2+u_2}{2}}{\frac{u_1+g_1}{2} - \frac{d_1+u_1}{2}}$$

$$= \frac{g_2-d_2}{g_1-d_1}.$$

The slope of segment DG (the third side of the triangle) is:

$$m_{DG} = \frac{g_2-d_2}{g_1-d_1} = m_{AB}.$$

Thus, segment AB (the mid-segment) is parallel to segment DG (the third side of the triangle).

Part (ii): If D and G lie on the same horizontal line, then $d_1 = g_1$, so the slope of segment DG is not defined. However, in this case, we know that both D and G lie on the line $y = d_1$.

The midpoints A and B are given by:

$$A = \left(\frac{d_1+u_1}{2}, \frac{d_2+u_2}{2}\right) \text{ and } B = \left(\frac{u_1+g_1}{2}, \frac{u_2+g_2}{2}\right).$$

Since $d_1 = g_1$, point B can be rewritten as:

$B = \left(\frac{u_1+d_1}{2}, \frac{u_2+g_2}{2}\right)$. Therefore, points A and B have the same x-coordinate, so A and B lie along the same horizontal line $y = \frac{u_1+d_1}{2}$. Since $y = d_1$ is parallel to $y = \frac{u_1+d_1}{2}$, we know that the mid-segment AB is parallel to the third side of the triangle DG.

Proof of Conjecture 2: The length of the mid-segment is half of the length of the third side.

Let A represent the midpoint of segment DU and let B represent the midpoint of segment UG. Then:

A = $\left(\frac{d_1+u_1}{2}, \frac{d_2+u_2}{2}\right)$ and B = $\left(\frac{u_1+g_1}{2}, \frac{u_2+g_2}{2}\right)$.

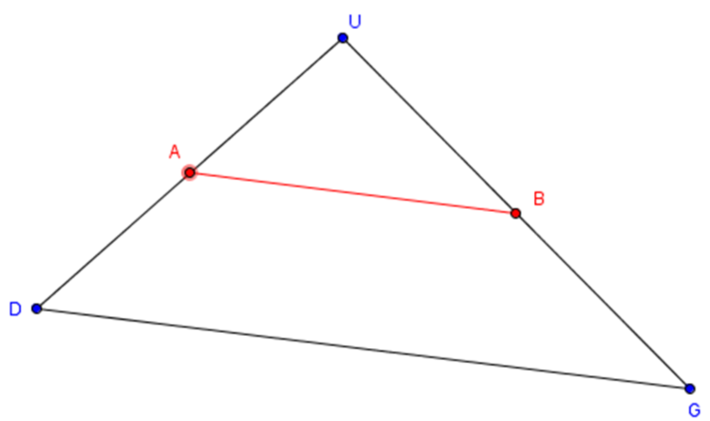
The length of segment DG is given by:

$$L_{DG} = \sqrt{(g_1-d_1)^2+(g_2-d_2)^2}$$

The length of the mid-segment is given by:

$$L_{AB} = \sqrt{\left(\frac{u_1+g_1}{2}-\frac{d_1+u_1}{2}\right)^2+\left(\frac{u_2+g_2}{2}-\frac{d_2+u_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(g_1-d_1)^2+(g_2-d_2)^2} \quad = \frac{1}{2} L_{DG}.$$



Possible Student Misconceptions	
Students may have difficulty with the calculations of the slope and the distance in parts (c) and (d).	How would these calculations work with the points you used in parts (a) and (b)? Why are the calculations more difficult in part (c)? (Teachers may need to remind students how to work with fractions and/or simplify roots.)
When proving the mid-segment is parallel to the third side, students may not think about the “third side” being horizontal when calculating the slope.	What happens if the mid-segment is on a horizontal line? Will your calculations still work?
Entry/Extensions	**Assessing and Advancing Questions**
If students can’t get started….	What does “midpoint” mean? How can you find the midpoint of one side? Does the segment connecting your midpoints look similar to the third side of the triangle? How? How can you compare these segments?
If students finish early….	Go back to the triangle in part (a) and draw the other two mid-segments. What do you notice about the three mid-segments? Is this property true for ALL triangles? How would you prove it? (Students should notice that putting in the three mid-segments should divide the original triangle into four congruent triangles.)

Discuss/Analyze

Whole Group Questions

Key understanding: The mid-segment is parallel to the third side of the triangle.

Questions:

- How did you calculate the slope of the mid-segment and the third side of the triangle in part (b)? Were the slopes the same no matter which two sides you chose to use to find the midpoints?
- Were the slopes the same for your mid-segment and third side in part (c) (no matter which two sides you chose to use to find the midpoints)?
- How did this exploration guide your proof in part (d)?

Key understanding: The length of the mid-segment is half of the length of the third side of the triangle.

Questions:

- How did you determine the length of the mid-segment was half of the length of the third side of the triangle in part (b)? Was this true no matter which two sides you chose to use to find the midpoints?
- Was this true in part (c) (no matter which two sides you chose to use to find the midpoints)?
- How did this exploration guide your proof in part (d)?

Midpoint Madness Task

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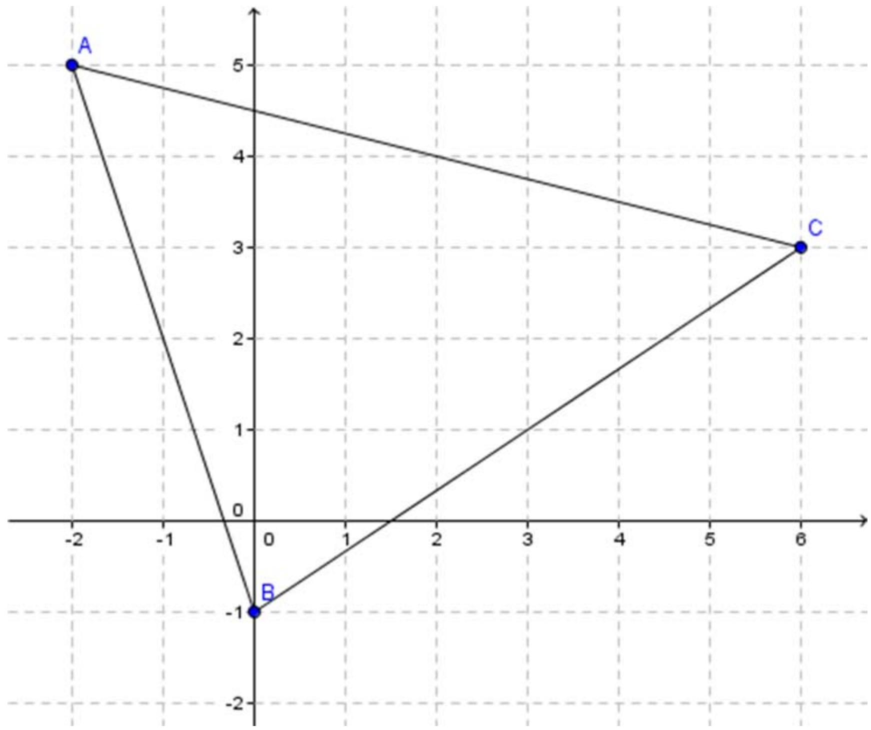
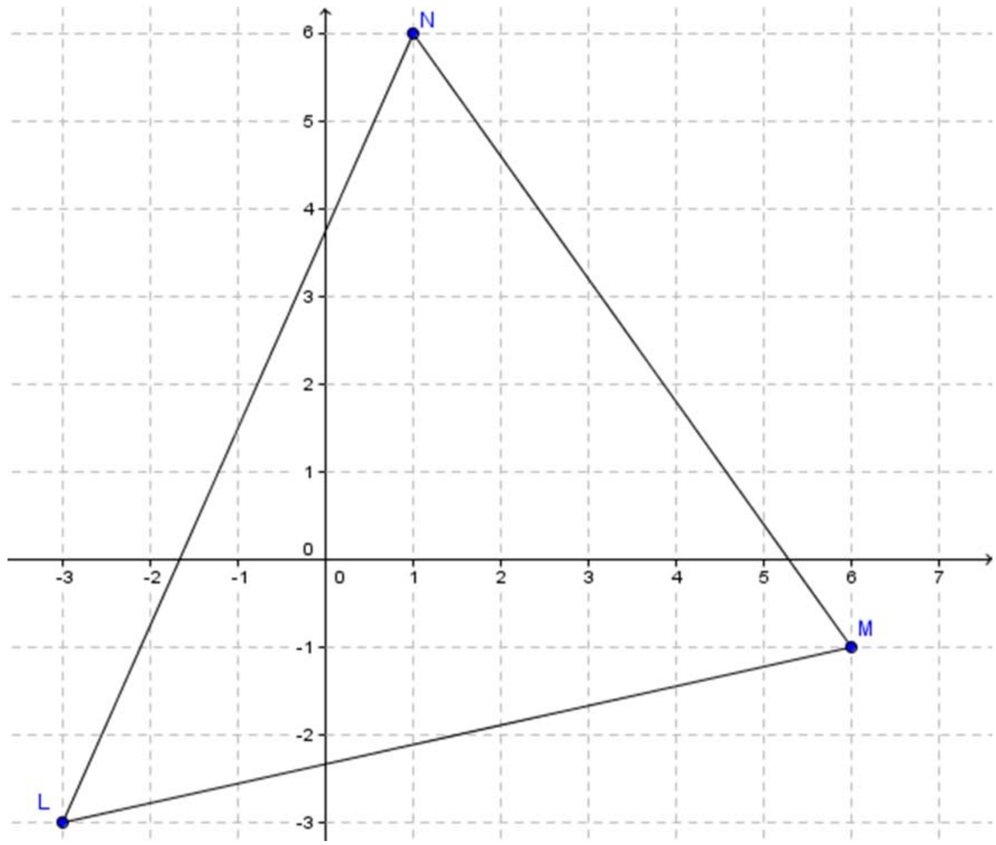


b) Draw the line segment connecting your midpoints from part (a). Compare this line segment to the third side of the triangle (the one without the midpoint marked). Make two conjectures about the relationship between the third side of the triangle and the line segment connecting the midpoints.

c) Repeat parts (a) and (b) on triangle LMN. Are your conjectures from part (b) true for this triangle? Explain how you know.



d) Are your conjectures true for every triangle? To find out, suppose triangle DUG is drawn on a coordinate plane with D located at point (d_1, d_2), U located at point (u_1, u_2), and G located at point (g_1, g_2). Write your conjectures in terms of D, U, and G, then prove your conjectures using these points.