Task: Making Right Triangles | **Geometry/Core Math II**

Line segment $\overline{AB}$ is drawn on the diagram below.

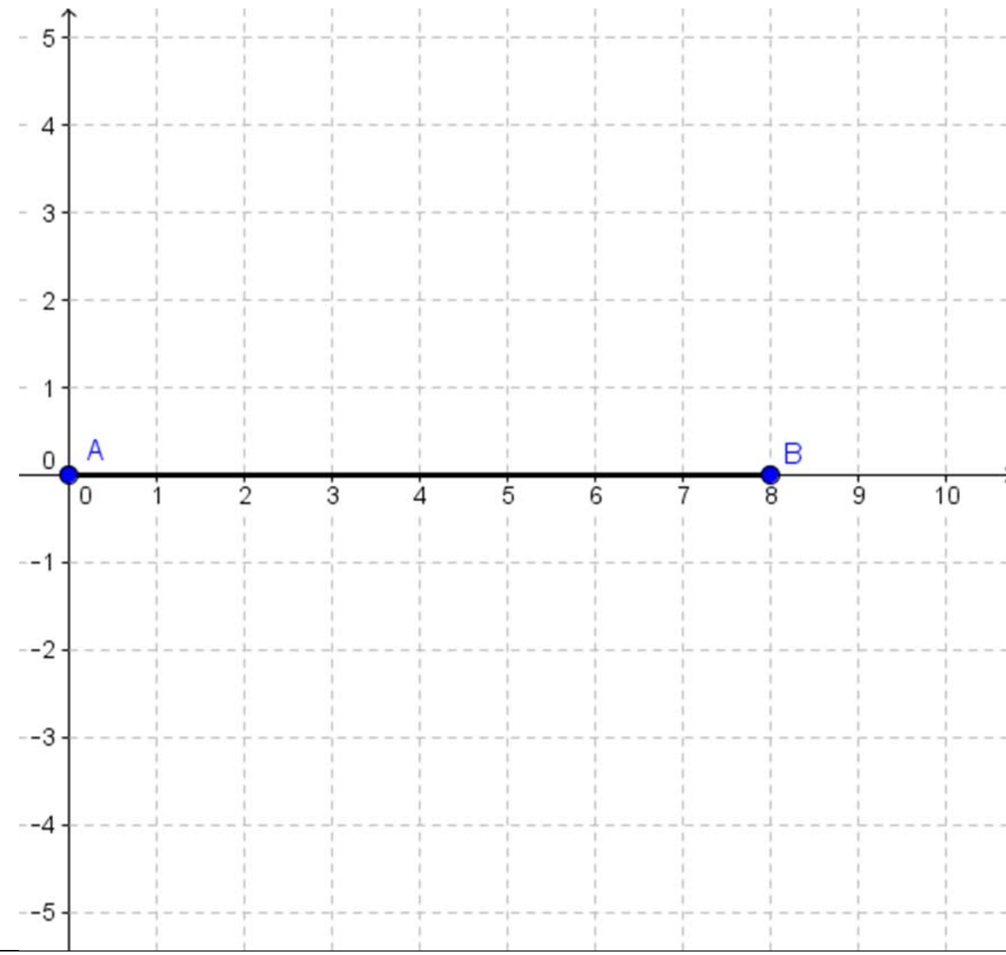
a) Locate a point C so that ABC is a right triangle with $m\angle ACB = 90°$ and the measure of one of the acute angles in the triangle is $45°$.

b) Locate a point D so that ABD is a right triangle with $m\angle ADB = 90°$ and the measure of one of the acute angles in the triangle is $30°$.

c) Locate a point E so that ABE is a right triangle with $m\angle AEB = 90°$ and the measure of one of the acute angles in the triangle is $15°$.

d) Find the distance between point C and the midpoint of segment $\overline{AB}$. Repeat with points D and E.

e) Suppose F is a point on the graph so that ABF is a right triangle with $m\angle AFB = 90°$. Make a conjecture about the point F. Explain why you think your conjecture is true.



Teacher Notes:

In this task, students are asked to find the coordinates of points in order to construct a right triangle. The given segment forms the hypotenuse of the triangle being constructed. Students must use trigonometric ratios to determine the coordinates of the point being sought.

Angles were chosen to provide “nice” answers. Teachers may want to change the values of the angles to show that the process applies no matter what angle is chosen.

Common Core State Standards for Mathematical Content	Common Core State Standards for Mathematical Practice
G-SRT.C.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.	Mathematical Practices 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

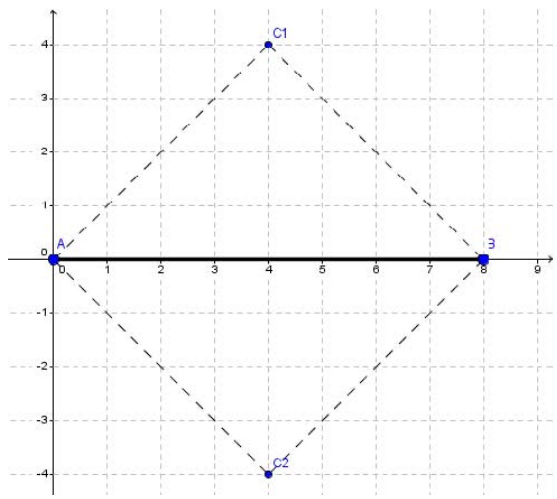
Essential Understandings

- A diagram is a “built” geometric artifact, with both a history—a narrative of successive construction—and a purpose.
- Underlying any geometric theorem is an invariance—something that does not change while something else does.
- Examining the possible variations of an invariant situation can lead to new conjectures and theorems.

Explore Phase

Possible Solution Paths

Part (a): Two possible positions for point C exist; these are labeled as C1 and C2 in the diagram. To find the x- and y-coordinates of C1, we need to first find the distance between A and C1, using triangle ABC1. We already know that the distance between A and B is 8 units.



We know that the measurement of one of the acute angles is $45°$; assume that this is the measurement of the angle with vertex A. For convenience, we will let w represent the distance between A and C1. Then:

$\cos 45° = \frac{w}{8}$, so $w = 8\cos 45°$. (Students may calculate this value, but here we will leave w as this expression so that we may use it in subsequent calculations without fear of introducing errors due to rounding.)

Our value of w tells us the distance between A and C1, but we do not know the exact coordinates of the point C1. We may graph the line through C1 and perpendicular to segment AB; this will create a new right triangle having a right angle whose vertex is on segment AB somewhere between the points A and B and with hypotenuse $\overline{AC1}$. (Note that the length of the hypotenuse is the value w calculated above.)

Assessing and Advancing Questions

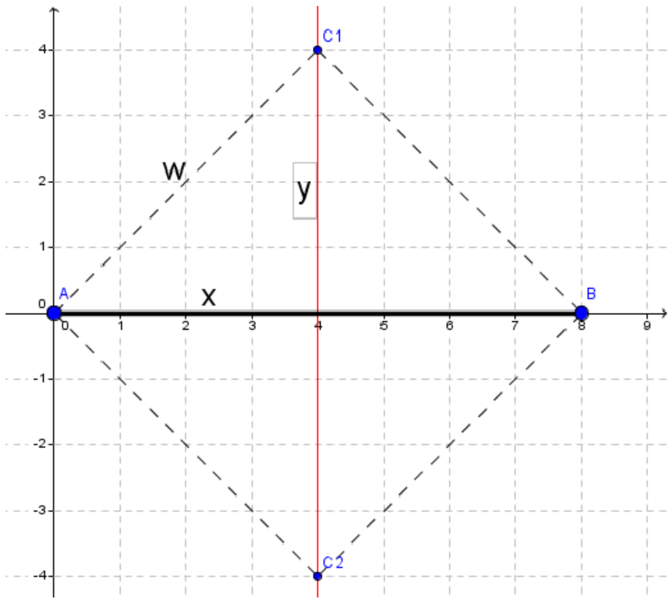
Assessing Questions:

How did you determine where C should be located?

How do you know that the angle with vertex C is a right angle?

Advancing Questions:

Sketch the triangle you want on a piece of scratch paper. Where is the acute angle? Where is the hypotenuse? How can you use this information to help you find where C should be located on your graph?

The 

coordinates of C1 can be found by calculating the values of x and y in the diagram above. Then:

$$\cos 45^\circ = \frac{x}{w}\text{, so } x = w\cos 45^\circ = (8\cos 45^\circ)\cos 45^\circ = 4$$

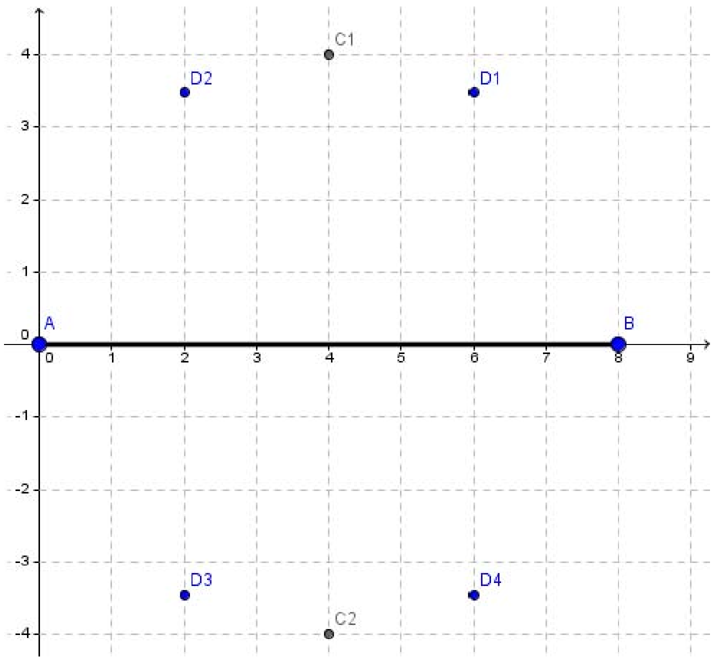
and

$$\sin 45^\circ = \frac{y}{w}\text{, so } y = w\sin 45^\circ = (8\cos 45^\circ)\sin 45^\circ = 4\,.$$

Students may also use similar triangles (comparing the smaller triangle we used to find the coordinates to the original triangle ABC1) to find the values of the coordinates. Doing so will result in evaluating the same calculations for the values of x and y.

Students may also know the trigonometric functions of certain “special angles,” including 45°, 30°, and 60° that can be used in the calculations in parts (a) and (b). If students have studied the half-angle formulas, they may opt to use those to do the calculations in part (c).

Part (b): Four possible positions for point D exist; these are labeled as D1, D2, D3, and D4 in the diagram. To find the x- and y-coordinates of D1, we need to first find the distance between A and D1, using triangle ABD1. We already know that the distance between A and B is 8 units.



We know that the measurement of one of the acute angles is $30°$; assume that this is the measurement of the angle with vertex A. For convenience, we will let w represent the distance between A and D1. Then:

$\cos 30° = \frac{w}{8}$, so $w = 8\cos 30°$. (Students may calculate this value, but here we will leave w as this expression so that we may use it in subsequent calculations without fear of introducing errors due to rounding.)

Our value of w tells us the distance between A and D1, but we do not know the exact coordinates of the point D1. We may graph the line through D1 and perpendicular to segment AB; this will create a new right triangle having a right angle whose vertex is on segment AB somewhere between the points A and B and with hypotenuse $\overline{AD1}$. (Note that the length of the hypotenuse is the value w calculated above.)

Assessing Questions:

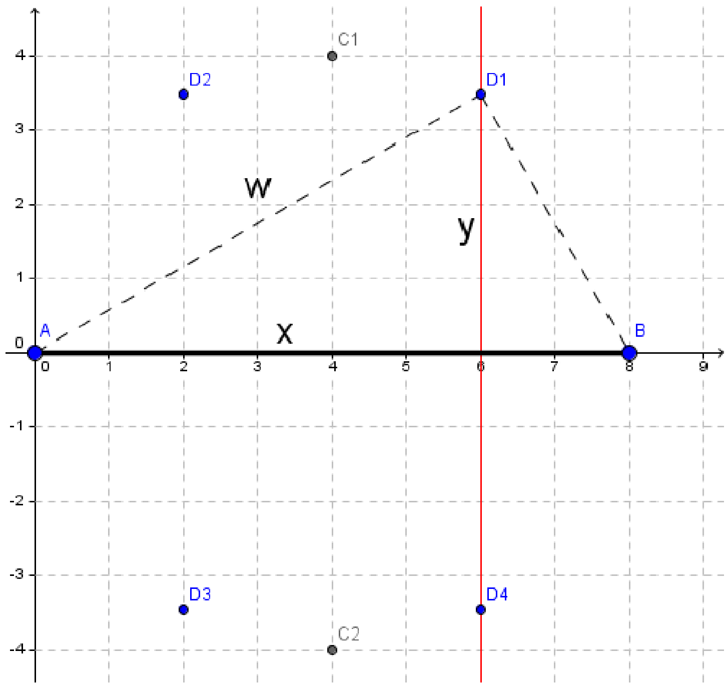
How did you determine where D should be located?

How do you know that the angle with vertex D is a right angle?

Are there other places where D could be located?

Advancing Questions:

Sketch the triangle you want on a piece of scratch paper. Where is the acute angle? Where is the hypotenuse? How can you use this information to help you find where D should be located on your graph?



The coordinates of D1 can be found by calculating the values of x and y in the diagram above. Then:

$$\cos 30^\circ = \frac{x}{w}, \text{ so } x = w\cos 30^\circ = \left(8\cos 30^\circ\right)\cos 30^\circ = 6$$

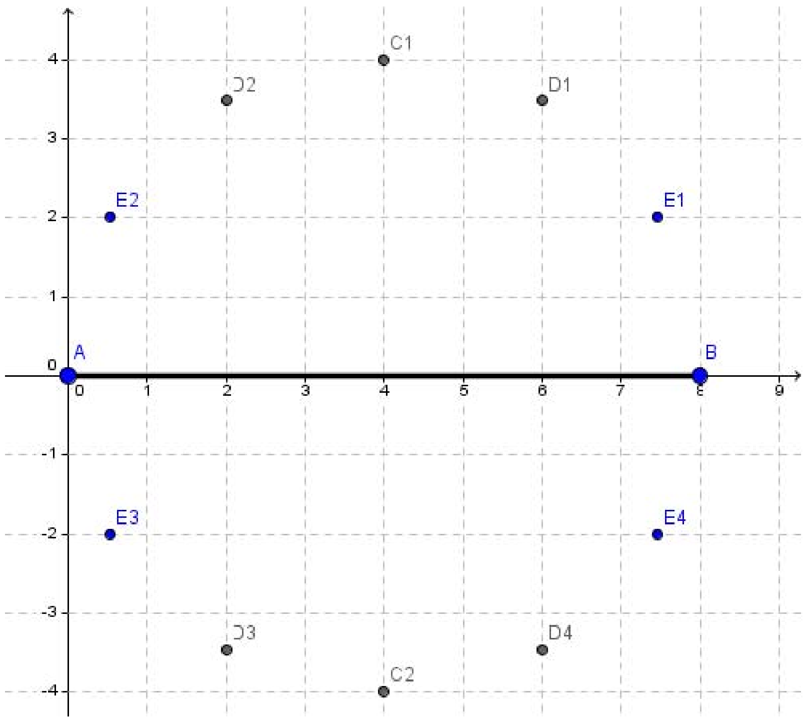
and

$$\sin 30^\circ = \frac{y}{w}, \text{ so } y = w\sin 30^\circ = (8\cos 30^\circ)\sin 30^\circ \approx 3.46.$$

Students may also use similar triangles (comparing the smaller triangle we used to find the coordinates to the original triangle ABD1) to find the values of the coordinates. Doing so will result in evaluating the same calculations for the values of x and y.

Students may also know the trigonometric functions of certain “special angles,” including 45°, 30°, and 60° that can be used in the calculations in parts (a) and (b). If students have studied the half-angle formulas, they may opt to use those to do the calculations in part (c).

Part (c): Four possible positions for point E exist; these are labeled as E1, E2, E3, and E4 in the diagram. To find the x- and y-coordinates of E1, we need to first find the distance between A and E1, using triangle ABE1. We already know that the distance between A and B is 8 units.



We know that the measurement of one of the acute angles is $15°$; assume that this is the measurement of the angle with vertex A. For convenience, we will let w represent the distance between A and E1. Then:

$\cos 15° = \frac{w}{8}$, so $w = 8\cos 15°$. (Students may calculate this value, but here we will leave w as this expression so that we may use it in subsequent calculations without fear of introducing errors due to rounding.)

Our value of w tells us the distance between A and E1, but we do not know the exact coordinates of the point E1. We may graph the line through E1 and perpendicular to segment AB; this will create a new right triangle having a right angle whose vertex is on segment AB somewhere between the points A and B and with hypotenuse $\overline{AE1}$. (Note that the length of the hypotenuse is the value w calculated above.)

Assessing Questions:

How did you determine where E should be located?

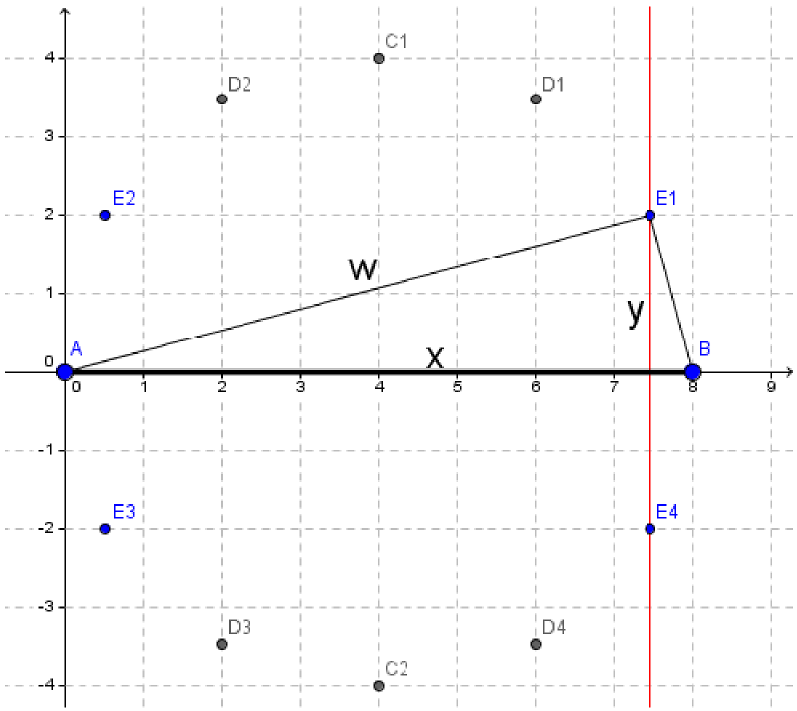
How do you know that the angle with vertex E is a right angle?

Are there other places where E could be located?

Advancing Questions:

Sketch the triangle you want on a piece of scratch paper. Where is the acute angle? Where is the hypotenuse? How can you use this information to help you find where E should be located on your graph?

Can you use your process from parts (a) and (b) to help you find the coordinates of point E?



The coordinates of E1 can be found by calculating the values of x and y in the diagram above. Then:

$$\cos 15^\circ = \frac{x}{w}, \text{ so } x = w\cos 15^\circ = (8\cos 15^\circ)\cos 15^\circ \approx 7.46$$

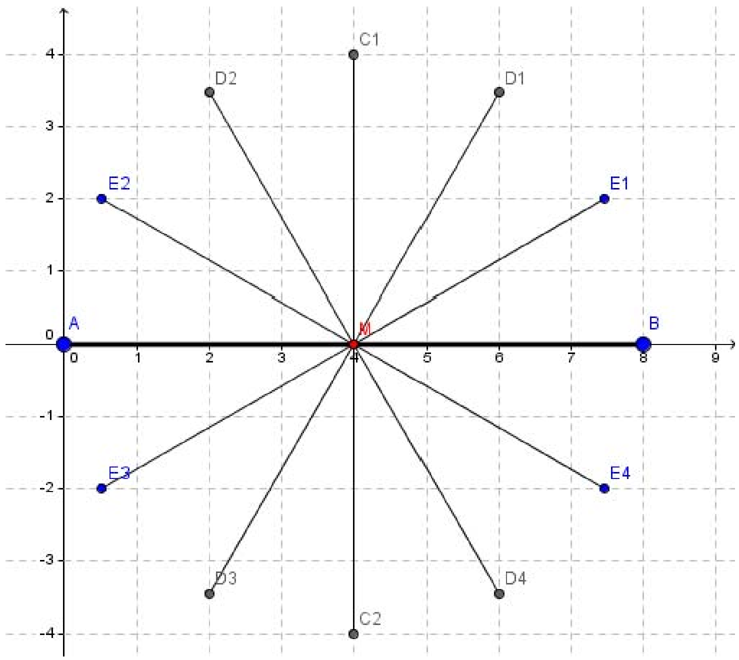
and

$$\sin 15^\circ = \frac{y}{w}, \text{ so } y = w\sin 15^\circ = (8\cos 15^\circ)\sin 15^\circ = 2.$$

Students may also use similar triangles (comparing the smaller triangle we used to find the coordinates to the original triangle ABE1) to find the values of the coordinates. Doing so will result in evaluating the same calculations for the values of x and y.

Students may also know the trigonometric functions of certain “special angles,” including 45°, 30°, and 60° that can be used in the calculations in parts (a) and (b). If students have studied the half-angle formulas, they may opt to use those to do the calculations in part (c).

Part (d): The distance between each of the points found in parts (a)-(c) and the midpoint of segment AB is 4. As an example, the distance between D1 and the midpoint is given; all other calculations follow the same pattern.



The midpoint of segment AB is (4, 0). Students may either use the midpoint formula or may simply count the distance on the graph. The coordinates of point D1 are (6, 3.46), with the y-coordinate rounded to two decimal places. Students may want to use the expression for y instead of the rounded value to avoid rounding error.

Then:

distance = $\sqrt{(6-4)^2+(3.46-0)^2} \approx 4$
(using the rounded value for y)

or

distance = $\sqrt{(6-4)^2+(8\cos 30^\circ \sin 30^\circ - 0)^2} = 4$
(using the expression for y).

Assessing Questions:

How did you find the midpoint of segment AB?

How did you calculate the distances between C, D, and E and the midpoint of segment AB?

Did you use exact values or rounded values for your coordinates in your calculations? How does using rounded values affect your calculations?

Advancing Questions:

Where is the midpoint of segment AB?

How do you find the distance between two points?

(If the student used rounded values for the coordinates) Would your distances change if you used more decimal places or exact values in your calculations? How?

Part (e): Conjecture: The point F will lie on a circle centered at (4, 0) with radius 4. This is based on the evidence we collected in part (d); since the distance between each of these points and the midpoint of the segment is 4, we know that the points we found in parts (a)-(c) all lie on a circle of radius 4.

Note: To prove the conjecture, students will need to use trigonometric identities, specifically the double angle formulae and $\sin^2\alpha + \cos^2\alpha = 1$. *At this point in the course, students may not have studied these identities. If students are familiar with these identities, the proof would be as follows:*

Let α *represent the acute angle in triangle ABF with vertex at A. Then, following the steps outlined in parts (a)-(c) above, we know that the x- and y-coordinates of F can be found by:*

$$x = \left(8\cos\alpha^\circ\right)\cos\alpha^\circ$$

and

$$y = (8\cos\alpha^\circ)\sin\alpha^\circ$$

(students would need to fill in the missing steps).

Then:

distance = $\sqrt{\left((8\cos\alpha)\cos\alpha - 4\right)^2 + \left((8\cos\alpha)\sin\alpha - 0\right)^2}$

$= \sqrt{\left(4\left(2\cos^2\alpha - 1\right)\right)^2 + \left(4\left(2\cos\alpha\sin\alpha\right)\right)^2}$

$= \sqrt{16\left(\cos(2\alpha)\right)^2 + 16\left(\sin(2\alpha)\right)^2}$

$= \sqrt{16(1)}$ *= 4.*

Since the distance between F and the midpoint of segment AB is always 4, and since this meets the definition of a circle, we know that F lies on the circle centered at (4, 0) with radius 4.

(Note that the cases where F is equal to either A or B should be handled separately in order to complete the circle.)

Assessing Questions:

What conjecture did you make? What evidence do you have to support your conjecture?

What would you need to do to prove your conjecture?

Advancing Questions:

Look at your distances you calculated in part (d). Do they have anything in common?

Do you think this property would be true for all points F as described in part (e)? Why or why not?

Possible Student Misconceptions	
Students may try to use segment AB as one leg of the triangle rather than as the hypotenuse.	What is the vertex of your right angle? What does this mean about segment AB?
Students may put the acute angle in the wrong place on the diagram.	Sketch a picture of your triangle on a piece of scratch paper. Where is the right angle? Where are the acute angles? How would this translate to your diagram?
Students may not realize that they need to use not only the right triangle they are trying to construct, but also a smaller right triangle to allow them to find the coordinates of their points.	When you found the distance between A (or B) and C (or D or E), how does that help you locate the position of C (or D or E)? Show me where the x-distance and the y-distance that you need are located on your diagram.
Students may use the wrong trigonometric function.	How do you find the sine of an angle? The cosine? On your diagram, where are the "opposite" and "adjacent" sides located in relation to your angle?
Students may round values in intermediate stages.	Did you round any values in the course of your calculations? Would this rounding affect the final answer? How?
Entry/Extensions	**Assessing and Advancing Questions**
If students can't get started….	How can you use the 45° angle to locate your point C? If one acute angle in your triangle has measure 45°, what would the measure of the other acute angle be? Would this help you find the location of C?
If students finish early….	Are there other locations for points C, D, and E? If so, where? Suppose segment AB was located somewhere else (perhaps as a vertical line segment, perhaps longer or shorter, perhaps on a diagonal). Would your conclusion be the same? Why or why not?

Discuss/Analyze

Whole Group Questions

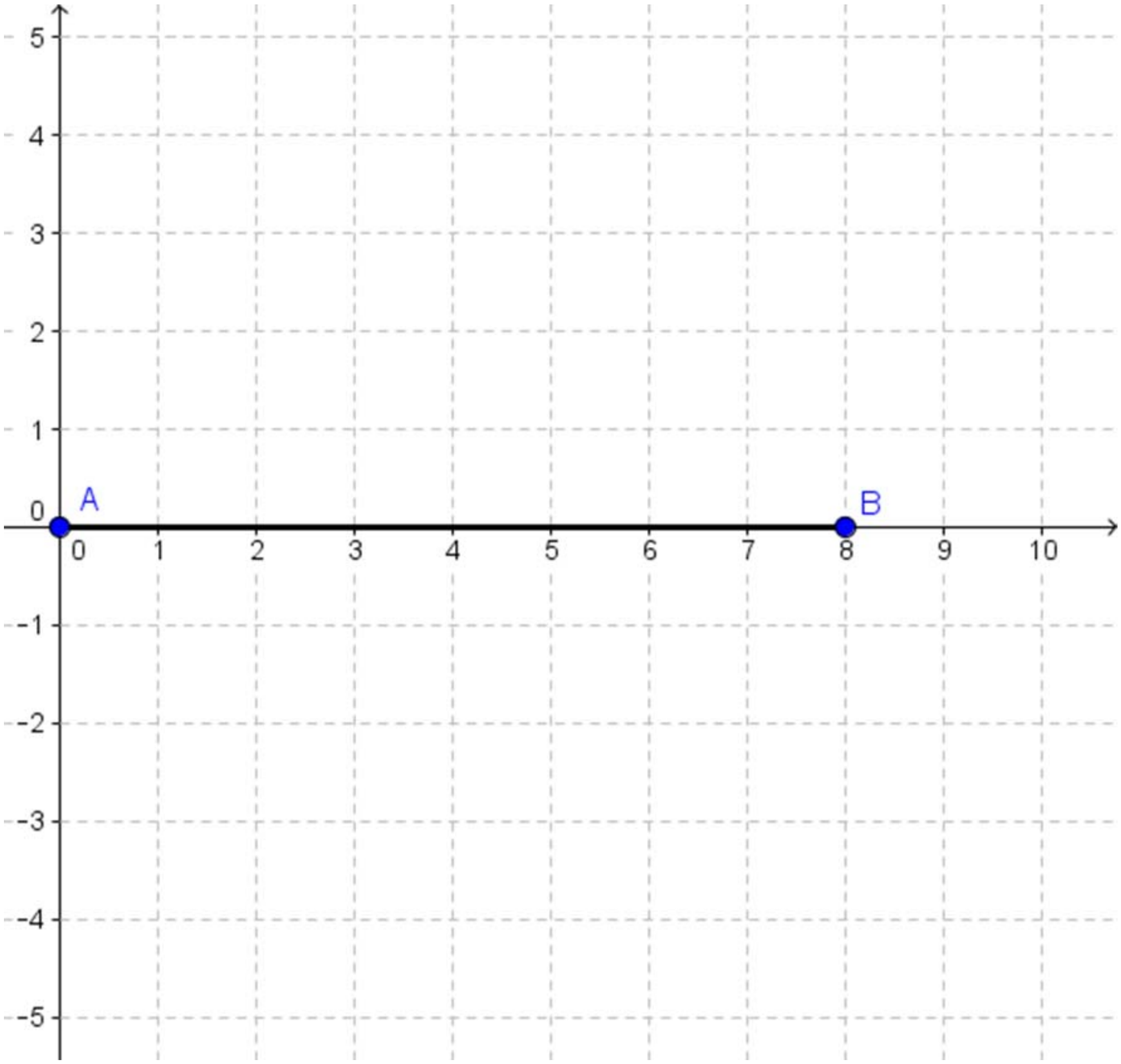
Key understanding: Given a particular line segment, many right triangles can be constructed so that that particular line segment forms the hypotenuse. The vertex of the right angle so constructed will always lie on a circle centered at the midpoint of the segment and radius equal to half of the length of the segment.

Questions:

- What did you discover about the distances between the points you found and the midpoint of the line segment?
- What conjecture did you make in part (e)? Do you think this conjecture is true for all values of the measurement of the acute angle?
- What do you think would happen if I made segment AB longer or shorter? How would the conjecture change?
- What would happen if I moved segment AB so that the segment was vertical instead of horizontal? What if the segment were "tilted"? Would your conjecture still be true? Why or why not?

Making Right Triangles

Line segment $\overline{AB}$ is drawn on the diagram below.



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c) Locate a point E so that ABE is a right triangle with $m\angle AEB = 90°$ and the measure of one of the acute angles in the triangle is $15°$.

d) Find the distance between point C and the midpoint of segment $\overline{AB}$. Repeat with points D and E.

e) Suppose F is a point on the graph so that ABF is a right triangle with $m\angle AFB = 90°$. Make a conjecture about the point F. Explain why you think your conjecture is true.