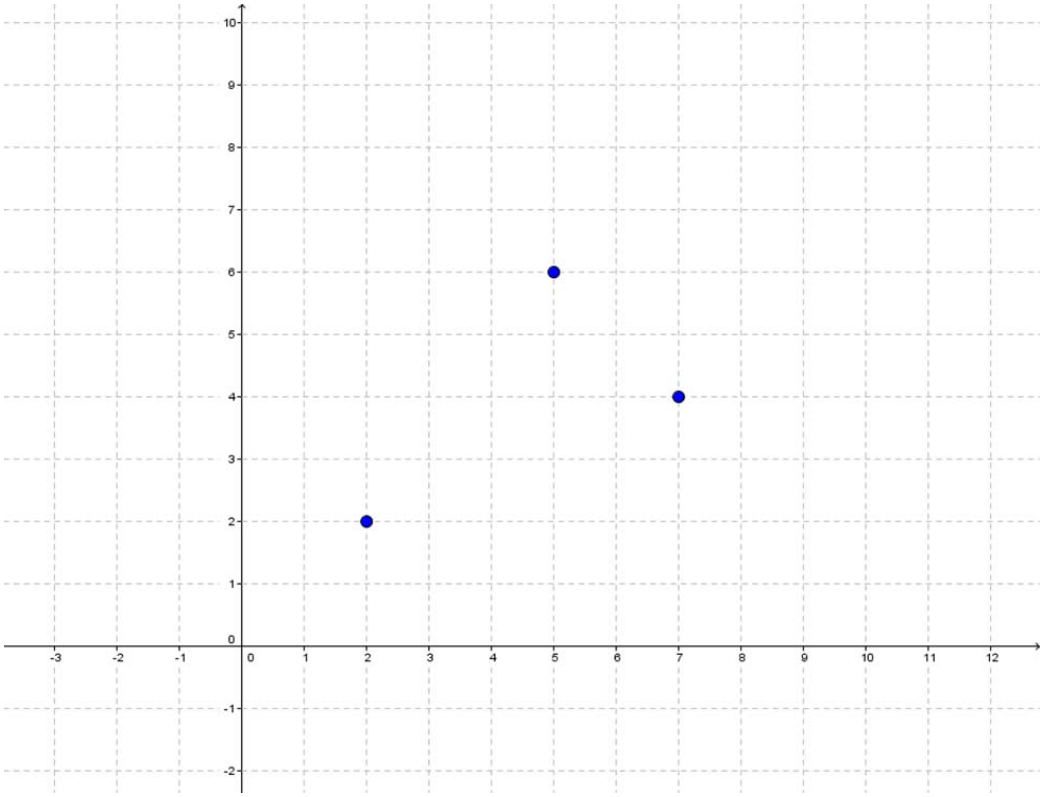
Task: Expanding Triangles **Geometry/Core Math III**

In the graph, three points are given. Label one point A, another point B, and the third point C.



a) Draw lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

b) Find a point D on $\overleftrightarrow{AB}$ such that A is the midpoint of $\overline{BD}$. Explain how you know A is the midpoint of $\overline{BD}$.

c) Find a point E on $\overleftrightarrow{AC}$ such that E is the midpoint of $\overline{CE}$. Explain how you know E is the midpoint of $\overline{CE}$.

d) Draw segments $\overline{BC}$ and $\overline{DE}$. What conjectures can you make about $\overline{BC}$ and $\overline{DE}$? Explain how you know your conjectures are true.

e) Make a conjecture about the quadrilateral BCDE. How do you know your conjecture is true?

Teacher Notes:

In this task, students will construct a quadrilateral and conclude that the quadrilateral constructed is a parallelogram. A reference to the "definition of a parallelogram" is made in the section describing solution paths; depending on when the task is used, students may use a theorem to prove that the quadrilateral constructed is a parallelogram rather than referring to a definition.

It is difficult to determine whether the wording quoted in G-CO 11 represents the hypothesis or the conclusion of the theorem being represented. While most class activities address theorems stated as "If quadrilateral BCDE is a parallelogram, then …..", this task takes the opposite approach. The goal of this task is for students to realize that "If quadrilateral BCDE has the properties that each pair of opposite sides are congruent and parallel, then quadrilateral BCDE is a parallelogram."

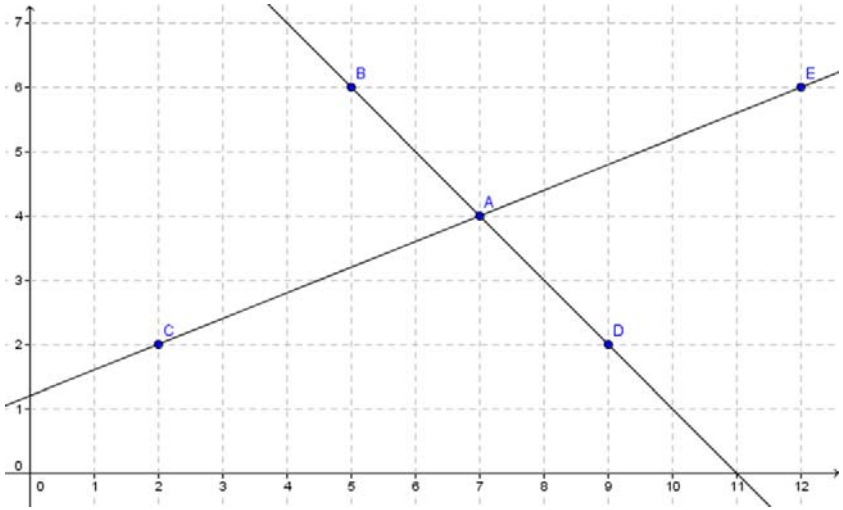
Common Core State Standards for Mathematical Content	Common Core State Standards for Mathematical Practice
G-CO.C.11 Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.* **G-GPE.B.4** Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).* **G-GPE.B.5** Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).	Mathematical Practices 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

Essential Understandings

- A diagram is a "built" geometric artifact, with both a history—a narrative of successive construction—and a purpose.
- Definitions in geometry are of two distinct types: definition *by genesis* (how you can create the object) and definition *by property* (how you can characterize the object in terms of certain features).

Explore Phase

Possible Solution Paths	Assessing and Advancing Questions
Parts (a), (b), and (c): There are six ways that students can label the points A, B, and C. These six different ways of labeling will lead to essentially three different diagrams, each resting on the initial choice of point A. These three diagrams are included below.	**Assessing Questions:** How did you determine where to place points D and E? **Advancing Questions:** How can you "move" along the grid lines from point A to point B? How can you use these "moves" to find point D? What does "midpoint" mean? How does that help you find points D and E?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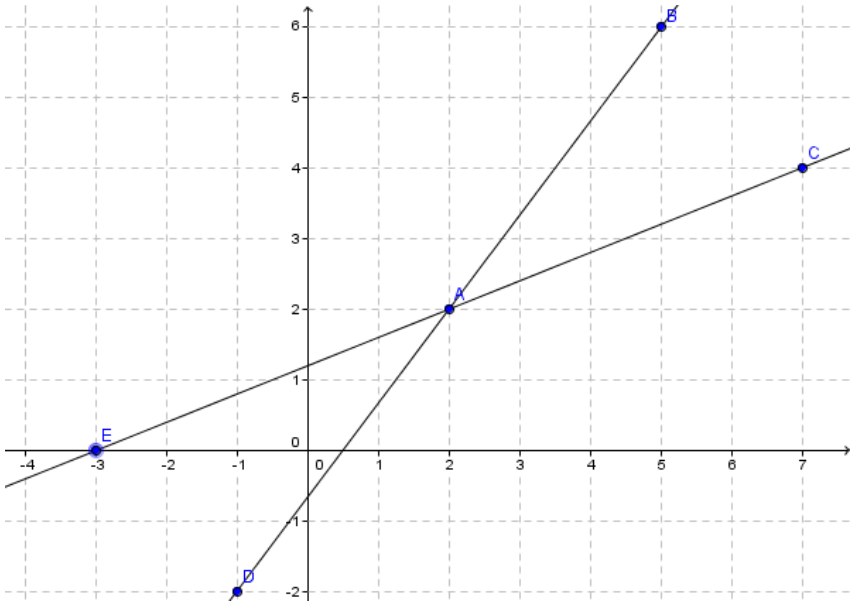
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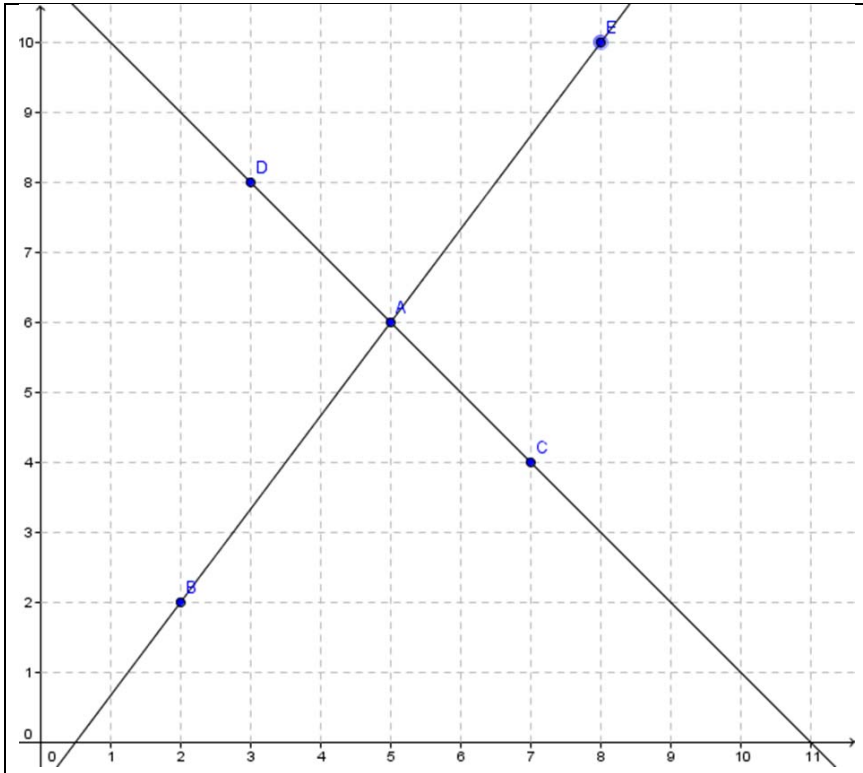


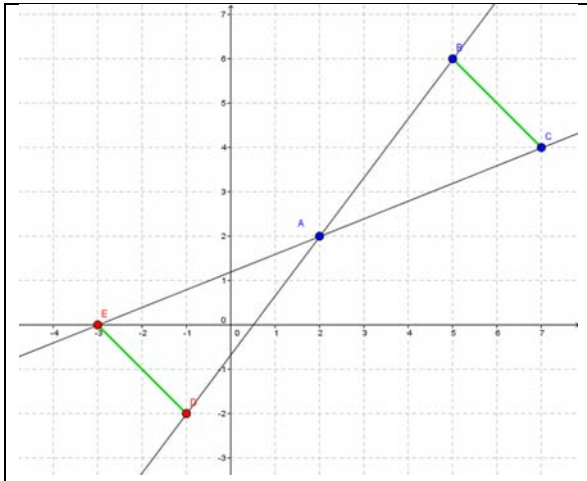
Part (d): The solution for part (d) is given for one diagram; the solutions for the other two diagrams are similar.

Assessing Questions:

What are your conjectures? How do you know your conjectures are true? Will your conjectures be true for any choices of A, B, and C? Why or why not?

Advancing Questions:

Compare segment BC to segment DE. What do they have in common? How would you show that this is true?



Conjecture 1: $\overline{BC}$ is parallel to $\overline{DE}$

To show this, students may calculate the slopes of $\overline{BC}$ and $\overline{DE}$.

The slope of $\overline{BC}$ is -1 (calculated by taking $\frac{4-6}{7-5}$).

The slope of $\overline{DE}$ is -1 (calculated by taking $\frac{-2-0}{-1-(-3)}$).

Conjecture 2: $\overline{BC}$ is the same length as $\overline{DE}$.

To show this, students may calculate the distances between the points.

The distance between B and C is:

$$\sqrt{(4-6)^2+(7-5)^2}=\sqrt{8}$$

The distance between D and E is:

$$\sqrt{(-2-0)^2+(-1-(-3))^2}=\sqrt{8}$$

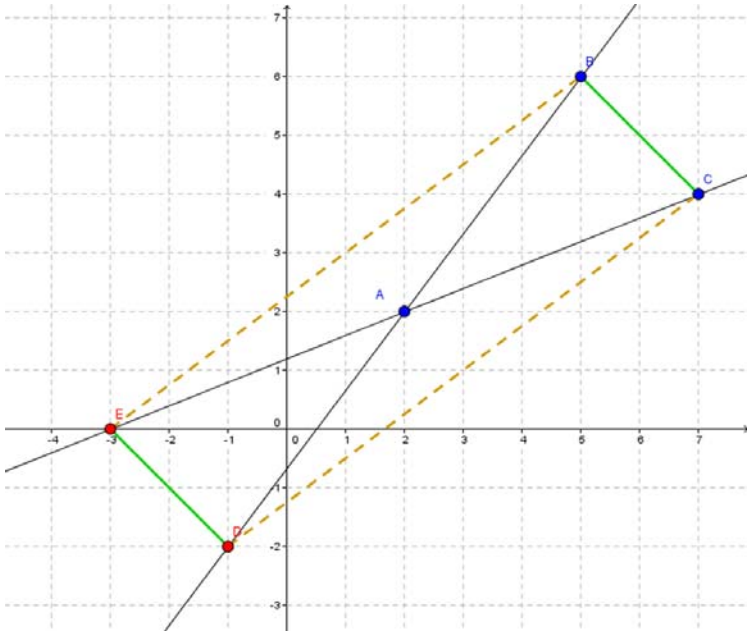
Students may also use congruent triangles (SAS) to show that these distances are equal.

Part (e): Conjecture: Quadrilateral BCDE is a parallelogram.

Assessing Questions:

In part (d), students proved that $\overline{BC}$ and $\overline{DE}$ are parallel and have the same length. By a similar argument, students can show that $\overline{BE}$ and $\overline{CD}$ are parallel and have the same length. Students may then use a definition for a parallelogram to conclude that quadrilateral BCDE is a parallelogram. Students may also see the diagonals as transversals and argue two congruent triangles exist within the parallelogram.

Note: For this particular configuration, students may assume that the quadrilateral is a rectangle. Although at first glance, this appears to be true, students should be able to check the slopes of two sides that share a common endpoint and conclude that the segments are not perpendicular. It would be interesting to ask where the point B can be relocated so that the resulting figure is a rectangle.



What did you conclude about quadrilateral BCDE? What did you need to show to "prove" your conjecture?

Advancing Questions:

Draw quadrilateral BCDE. What does quadrilateral BCDE look like to you? How would you "prove" that this is true?

Possible Student Misconceptions

Students may put points D and E in the wrong place.
Students may not know how to prove whether two lines are parallel or perpendicular.

What does "midpoint" mean? Did you check the distance between A and D and compare it to the distance between A and B? Is D on line AB? (Similar questions can be asked for point E.)

Entry/Extensions

Assessing and Advancing Questions

If students can't get started….	**Advancing Questions:** How can you "move" along the grid lines from point A to point B? How can you use these "moves" to find point D? What does "midpoint" mean? How does that help you find points D and E?
If students finish early….	Use the triangles in your diagram to show that opposite angles in the parallelogram are congruent. Where could you relocate the point B in order to make quadrilateral BCDE a rectangle when you go through these steps? Can you always move B to form a rectangle no matter where A and C are located?

Discuss/Analyze

Whole Group Questions

Key understanding: Quadrilateral BCDE, as constructed, is a parallelogram.

Questions:

- How did you "prove" that quadrilateral BCDE is a parallelogram? *(Note: Students may use either the distance formula or appeal to congruent triangles to show that opposite sides are congruent.)*
- Is quadrilateral BCDE ALWAYS a parallelogram, no matter where A, B, and C are located? *(Note: If A, B, and C are collinear, then quadrilateral BCDE "collapses" into a line segment.)*
- How could you use your diagram to show that opposite angles are congruent?

Expanding Triangles Task

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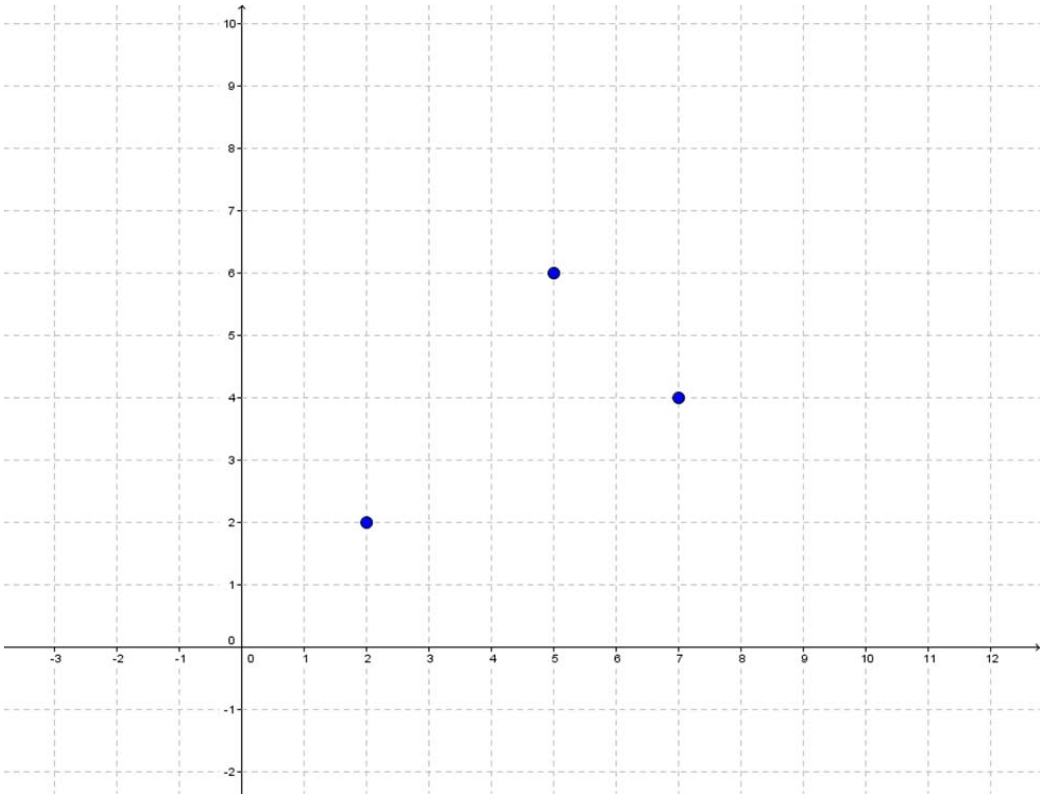

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