Task: Culture Shock **Algebra II**

The populations of two bacterial cultures after 0 hours, 1 hour, 2 hours, and 3 hours are given in the table below.

Hours	Culture A	Culture B
0	1000	1
1	2000	4
2	4000	16
3	8000	64

Assume the pattern of growth continues for both culture A and culture B.

a) Find an expression to calculate the population of culture A after n hours. Explain your reasoning.

b) Find an expression to calculate the population of culture B after n hours. Explain your reasoning.

c) Which culture will have the greatest population after 8 hours? How do you know?

d) Will the population of culture A ever equal the population of culture B? If so, when? If not, why not?

e) As time passes, which will have a bigger influence on the population of the culture: the starting population or the growth rate of the culture? Why?

Teacher Notes:

This task focuses on geometric sequences and the influence of the starting term and the common ratio on the terms of the sequence.

Common Core State Standards for Mathematical Content	Common Core State Standards for Mathematical Practice
(F-BF) Build a function that models a relationship between two quantities 2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★	Mathematical Practices 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics.

(A-SSE) Write expressions in equivalent forms to solve problems
3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★
c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Essential Understandings

The concept of function is intentionally broad and flexible, allowing it to apply to a wide range of situations. The notion of function encompasses many types of mathematical entities in addition to "classical" functions that describe quantities that vary continuously. For example, matrices and arithmetic and geometric sequences can be viewed as functions.

Functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, and tabular representations. Links among these different representations are important to studying relationships and change.

Exponential functions are characterized by a rate of change that is proportional to the value of the function. It is a property of exponential functions that whenever the input is increased by 1 unit, the output is multiplied by a constant factor.

Explore Phase

Possible Solution Paths

a) Students may approach the problem using either a recursive formula or a general formula. In either case, students should use an appropriate approach to a geometric sequence. Note that the problem is worded so that the first term of the geometric sequence is actually labeled a_0, where the subscript represents the number of hours.

Recursive approach:

Assume $a_0 = 1000$. Then:

$a_1 = 2000 = 2(1000) = 2(a_0)$

$a_2 = 4000 = 2(2000) = 2(a_1)$

…

$a_n = 2(a_{n-1})$

Assessing and Advancing Questions

Assessing:

How did you describe your pattern?

How did you use the information in the table to create your formula?

Does your formula work with all of the values in the table for culture A?

Advancing:

Do you see a pattern in the table?

How can that pattern help you create a formula?

General formula approach: Assume $a_0 = 1000$. Then: $a_1 = 2000 = 2\ (1000) = 2(a_0)$ $a_2 = 4000 = 2\ (2000) = 2(2(a_0)) = 2^2\ (a_0)$ … $a_n = 2^n\ (a_0)$	
b) Again, students may approach the problem using either a recursive formula or a general formula. In either case, students should use an appropriate approach to a geometric sequence. Note that the problem is worded so that the first term of the geometric sequence is actually labeled b_0, where the subscript represents the number of hours. *Recursive approach:* Assume $b_0 = 1$. Then: $b_1 = 4 = 4\ (1) = 4(b_0)$ $b_2 = 16 = 4\ (4) = 4(b_1)$ … $b_n = 4(b_{n-1})$ *General formula approach:* Assume $b_0 = 1$. Then: $b_1 = 4 = 4\ (1) = 4(b_0)$ $b_2 = 16 = 4\ (4) = 4(4(b_0)) = 4^2\ (b_0)$ … $b_n = 4^n\ (b_0)$	**Assessing:** How did you describe your pattern? How did you use the information in the table to create your formula? Does your formula work with all of the values in the table for culture B? **Advancing:** Do you see a pattern in the table? How can that pattern help you create a formula?
c) Students may either extend the table given in the problem (which is	**Assessing:**

equivalent to using the recursive formulas developed in parts (a) and (b)) or may use the general formulas if those were developed in parts (a) and (b). Students may also graph the general formulas for the populations and compare when x = 8.

Using the recursive formula to extend the table:

Hours	Culture A	Culture B
0	1000	1
1	2000	4
2	4000	16
3	8000	64
4	16000	256
5	32000	1024
6	64000	4096
7	128000	16384
8	256000	65536

Using the general formula:

For culture A, the population after n = 8 hours is:

$a_8 = 2^8 (a_0) = 2^8 (1000) = 256000$.

For culture B, the population after n = 8 hours is:

$b_8 = 4^8 (b_0) = 4^8 (1) = 65536$.

After 8 hours, culture A still has a higher population.

Graph:

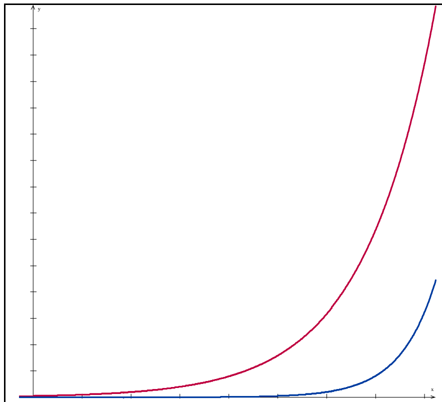
How did you determine the populations in Hour 8?

Are there other ways to determine this? Can these other ways be used to check your work?

Advancing:

When do we want to know the population (at what hour)?

Can we use the pattern (or formula) we identified in parts (a) and (b) to extend our table?

 In this graph, the red represents culture A and the blue represents culture B. Due to the size of the populations, the numbers on the axes are difficult to read, but the student should get the general idea that culture A's population will exceed culture B's population when the number of hours is 8. (Here, the x-axis runs from -1 to 8.5.)	
d) Students may answer the question regarding whether the populations will ever be equal in multiple ways. For example, students may extend the table to n = 9 and n = 10 hours and notice that when n = 10, culture B's population exceeds culture A's population. Because culture A's population exceeded culture B's population when n = 9, students should conclude that at some point between n = 9 and n = 10 hours, the two populations must be equal. To determine WHEN the populations are equal, students may use a graphing calculator or may use the general formulas developed in parts (a) and (b). *Using the general formulas*, most students will likely elect to set the two expressions equal and solve for the value of n: $2^n (1000) = 4^n$ There are two approaches to solving this equation once it is set up:	**Assessing:** How did you decide whether the two populations would ever be equal? How did you find when the populations were equal? **Advancing:** Do you think the populations will ever be equal? Why or why not? [If students do not think the populations will ever be equal]: Can you use the ideas you developed to answer part (c) to explain why the populations will not be equal? [If students think the populations will be equal]: How can you use the ideas you developed in part (c) to find when the populations are equal?

Approach 1: Recognize that $4 = 2^2$ and substitute, then use a property of exponents: $2^n (1000) = (2^2)^n$ $1000 = \frac{(2^2)^n}{2^n} = \frac{2^{2n}}{2^n} = 2^n$ Approach 2: Divide both sides by 2^n, then use a different property of exponents: $2^n (1000) = 4^n$ $1000 = \frac{4^n}{2^n} = \left(\frac{4}{2}\right)^n = 2^n$ To solve the equation $1000 = 2^n$, students will most likely use logarithms, but they may choose among three different bases. Using ln: $\ln (1000) = \ln (2^n)$ $\ln (1000) = n \ln (2)$ $\frac{\ln(1000)}{\ln(2)} = n$ Similar calculations occur if students elect to use log rather than ln. In either of these cases, students can use their calculator to evaluate the value of n and get: $n = 9.97$ years (rounded to two decimal places).	

Students may also elect to use $\log_2$ rather than ln or log:

$$\log_2 (1000) = \log_2 (2^n)$$

$$\log_2 (1000) = n \log_2 (2)$$

$$\frac{\log_2(1000)}{\log_2 (2)} = n$$

$$\log_2 (1000) = n \quad (\text{since } \log_2 2 = 1)$$

In this case, students must now use the change of base formula

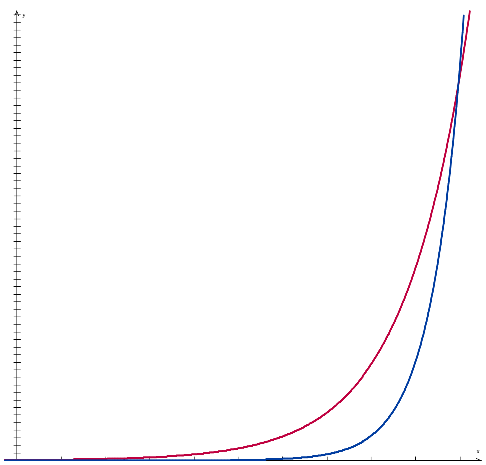
$$\left(\log_a b = \frac{\ln b}{\ln a} = \frac{\log b}{\log a}\right)$$

to calculate the value of n.

Another approach that students may use is to estimate the value of n that will satisfy $2^n = 1000$ by using a guess and check approach with tenths, then hundredths. The table below illustrates the approach, but only the relevant values are included to give an idea of what would happen.

n	2^n	Conclusion (compare to 1000)
9	512	Too low
10	1024	Too high—need a number between 9 and 10
9.9	955.42	Too low—need a number between 9.9 and 10
9.96	995.999	Too low—need a number between 9.96 and 10
9.97	1002.93	Too high—need a number between 9.96 and 9.97
etc.		

Graph:



In this graph, the red represents culture A and the blue represents culture B. Due to the size of the populations, the numbers on the axes are difficult to read, but the student should get the general idea that culture B's population will eventually exceed culture A's population. (The x-axis runs from -1 to 10.5 here.) Students can use the "intersect" option on their TI calculators to calculate the intersection point.

e) Early in the analysis, students may believe that the starting term has a larger influence. Culture A began with a population of 1000 and culture B began with a population of 1, and in the early stages of exploration (through part (c)), it was clear from the graph and the numbers in the table that population of culture A continued to exceed the population of culture B by a large margin. In fact, the graph shown in part (c) above makes it appear that the curves are maintaining a significant distance apart as the number of hours increases. However, in part (d), students realize that the population of culture B "catches up" with and passes the population of culture A since, in each hour, the population of culture B is multiplied by a factor of 4 while the population of culture A is only multiplied by a factor of 2. It simply took a while for the population of culture B to catch up. This is also evident in the graph shown in part (d); the curve associated with culture B is much steeper than the curve associated with culture A.	**Assessing:** Which term has the largest influence—the starting population or the growth rate? Why? [If students believe that the growth rate has the largest influence]: Another student thinks that the starting population has the largest influence. How would you explain your answer to him? **Advancing:** In part (d), you saw that the population of culture B eventually exceeded the population of culture A. What caused the population of culture B to "catch up" to the population of culture A? Why?
Possible Student Misconceptions	
Parts (a) and (b) (carries over to parts (c) and (d)): Students are used to seeing the n^{th} term of a geometric sequence written as $a_n = a_1 r^{(n-1)}$, where a_n is the n^{th} term, a_1 is the first term, r is the common ratio, and n is the number of terms. In this case, the population of each of the cultures begins with hour n = 0, so the first term is a_0, not a_1. This could cause all their calculations using n to be "off by 1." (So, for example, their expression for the population of culture A at hour n would be calculated using $(1000)\ 2^{n-1}$.) By beginning with a_0 instead of a_1, their exponent on the common ratio should be n, not n – 1. (Note: This misconception, if allowed to continue, will also carry over to parts (c) and (d) as well.)	**Assessing:** Does your formula work with the numbers given in the table? **Advancing:** What would be the next line in the table? Does your formula work with that set of numbers as well?
Part (c): If students are "off by 1" in parts (a) and (b), their answer to part (c) is	Go back and ask questions about parts (a) and (b).

the same (the population of culture A still exceeds the population of culture B), but the precision is incorrect.	
Part (c)—graphing option: Students may use the standard window (-10 to 10 in both directions) to graph.	**Assessing:** Is your graph showing up? What should you do to see the rest of the graph? **Advancing:** Is there any need to include negative values of x in your graph?
Part (d)—properties of exponents: Students may have difficulty applying the properties of exponents correctly: $\frac{4^n}{2^n} = (4-2)^n = 2^n$ OR $\frac{4^n}{2^n} = 4^n - 2^n = 2^n$ OR $\frac{4^n}{2^n} = 4^n - 2^n = (4-2)^n = 2^n$ (Note: All three of these will result in the “correct answer” but through a process of incorrect reasoning.) OR $(2^2)^n = 2^{2+n}$ OR	**Assessing:** What property of exponents are you using? Is the property you are applying appropriate? **Advancing:** Have you applied the property correctly? How can you tell?

$\frac{4^n}{2^n} = \left(\frac{4}{2}\right)^{2n} = 2^{2n}$	
Part (d)—properties of logarithms: Students may have difficulty correctly applying the appropriate property of logarithms (note: ln may be replaced by other log bases in these examples): ln (2^n) = (ln 2) (ln n) OR ln (2^n) = 2 ln n OR $\frac{\ln(1000)}{\ln(2)} = ln\left(\frac{1000}{2}\right) = \ln(500)$	**Assessing:** What property of logarithms are you using? Is the property you are applying appropriate? **Advancing:** Have you applied the property correctly? How can you tell?
Part (d)—calculating logarithms: Students may have difficulty putting the logarithms into their calculator (note: ln may be replaced by other log bases): $\frac{\ln(1000)}{\ln(2)}$ is input as ln (1000 / ln (2) OR $\frac{\log_2(1000)}{\log_2(2)}$ is input as log (2(1000) / log (2(2)) OR	**Assessing:** Do you need any parentheses in your calculation? Where? **Advancing:** For the base 2 calculations: How do you get the calculator to calculate $\log_2$ of something? Is $\log_2$ "built into" your calculator?

<table>
<tr><td>if students do not recognize that $\log_2 2 = 1$:

$\frac{\log_2 (1000)}{\log_2 (2)}$ is input as (log(1000 / log (2)) / (log (2 / log (2))</td><td></td></tr>
<tr><td>Part (d)—change of base issues:

Students may not know how to correctly apply the change of base formula:

$\log_2 (1000) = \log (2) / \log (1000)$</td><td>Assessing:
What is the change of base formula?

What are the values of the variables in the change of base formula?</td></tr>
<tr><td>Entry/Extensions</td><td>Assessing and Advancing Questions</td></tr>
<tr><td>If students can’t get started….</td><td>Do you see any patterns in the table? How can the patterns help you with finding a formula?</td></tr>
<tr><td>If students finish early….</td><td>What effect does the initial population of each culture have on how long it takes the population of culture B to exceed the population of culture A?

Suppose the initial population of culture B was 5, and the growth rate was the same. What effect would this have on your calculations in parts (c) and (d)? Why?

OR

Suppose you have a culture C with the populations given in the table below:
<table>
<tr><th>Hours</th><th>Culture C</th></tr>
<tr><td>0</td><td>1,000,000</td></tr>
<tr><td>1</td><td>1,500,000</td></tr>
<tr><td>2</td><td>2,000,000</td></tr>
<tr><td>3</td><td>2,500,000</td></tr>
</table>
Compare the growth of culture A and culture B with culture C. What can you tell me about how these populations compare? Will</td></tr>
</table>

<table>
<tr><td></td><td>the population of culture A and/or culture B ever pass the population of culture C? Why or why not?</td></tr>
<tr><td colspan="2">Discuss/Analyze</td></tr>
<tr><td colspan="2">Whole Group Questions</td></tr>
<tr><td colspan="2">Key understanding: The growth rate of a geometric sequence has a bigger effect on the eventual size of terms than the initial term of the sequence.

This is exactly the point of part (e), so those questions apply here.</td></tr>
<tr><td colspan="2">Key understanding: A geometric sequence can be viewed as a function, and processes associated with analyzing a function can be used to analyze a geometric sequence.

Questions:
Did any of you graph the geometric sequences? What did you discover with your graphs?

In part (d), you had to determine when the populations were equal. You discovered that the time at which the populations were equal was not a whole number of hours. How does this fit with your notion of a geometric sequence?</td></tr>
<tr><td colspan="2">Key understanding: A function can be represented in multiple ways.

Questions:
How did we represent the function as we worked through this problem?

How does each of these representations give us information about the function?</td></tr>
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